Unveiling complexity of church bells dynamics using experimentally validated hybrid dynamical model

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Abstract. We investigate the dynamics of the yoke-bell-clapper impacting system using a novel hybrid dynamical model, which is verified experimentally by comparing the results of numerical simulations with experimental data obtained from the biggest bell in the Cathedral Basilica of St Stanislaus Kostka, Lodz, Poland. Having proved that the model is a reliable predictive tool we present a plethora of different dynamical behaviours that can be observed in the considered system. We highlight the solutions that can be considered as a proper working regimes of the instrument and describe how to obtain them. Detailed bifurcation analysis allow us to present how the design of the yoke and propulsion mechanism influence the response of the system. Presented results prove the feasibility of the developed model and demonstrate the importance of nonlinear analysis in practical engineering applications.
Keywords: Bells dynamics, Bifurcation analysis, Impacting system, Hybrid system, Computer simulations, Experimental verification.

1 Introduction

Bells are musical instruments that are closely connected with European cultural heritage. Although the design of a bell, its clapper and a belfry has been developed for centuries, mathematical modeling of their behaviour have been encountered recently. It is surprising from an engineering point of view because bells and their supports are structures that are exposed to severe loading conditions during ringing. To ensure that they will work reliably for ages, we have to consider many factors and one of the most important is to ensure safe and effective operation. Moreover, the dynamics a yoke-bell-clapper system is extremely complex and difficult to analyze due to its nonlinear characteristic, repetitive impacts and complicated excitation.

The first attempt to describe bell's behaviour using equation of motion was made by Veltmann in 19th century [1,2]. His work was stimulated by the

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failure of the famous Emperor's bell in the Cologne Cathedral when the clapper remained always on the middle axis of the bell instead of striking it. Veltmann explained the reason of this phenomena using simple model developed basing on the equations of a double physical pendulum. Heyman and Threlfrall [3] use similar model to estimate inertia forces induced by a swinging bell. The knowledge of loads induced by ringing bells depends on the mounting layout and is crucial during the design and restoration processes of belfries. In Europe one can distinguish three different types of swinging bells: Central European, English and Spanish. In the first of them, bells tilt on their axis and maximum amplitude of oscillation is usually below 90 degrees. In the English system bells perform nearly a complete rotation (the bell stops close to upper position); while in the Spanish system bells rotate continuously in the same direction.

Muller [4] and Steiner [5] analyze the dynamic interactions between bells and bell towers and describe the dynamic forces appearing in bells mounted in Central European manner. There are also similar studies concerning English system [6,7] as well as Spanish system [8,9]. Ivorra et. al. [10,11] show how mounting layout affects the dynamic forces induced by bells. Authors prove that in the Spanish system forces transmitted to the supporting structure are significantly lower than in English and Central European. Nevertheless, studies described in [12,13] prove that often minor modification in support's design can significantly decrease the probability of damage of the bell tower.

Klemenc et. al. contributed with a series of papers [14,15] devoted to the analysis of the clapper-to-bell impacts. Presented results prove that full-scale finite-element model is able to reproduce the effect of collisions but requires long computational times and complex, detailed models. Therefore, it would be difficult to use such models to analyze the dynamics of bells. Because of that, recently we observe the tendency to use hybrid dynamical models which are much simpler and give accurate results with less modeling and computational effort. In [16] authors propose lumped parameter model of the bells mounted in Central European system and prove that with the model we are able to predict impact acceleration and bell's period of motion.

The paper is organized as follows. In Section 2 we describe the hybrid dynamical model of the church bell and validate it by comparing the results of numerical simulations with data obtained experimentally. In Section 3 we characterize the 7 most common working regimes and in Section 4 investigate how they can be obtained. Finally, in Section 5 the conclusions are drawn.

2 Model of the system

The model that is presented in this paper is build up based on the analogy between freely swinging bell and the motion of the equivalent double physical pendulum. The first pendulum has fixed axis of rotation and models the yoke and the bell that is mounted on it. The clapper is imitated by the second pendulum attached to the first one. Proposed model involves eight physical parameters. The photo of the bell that has been measured to obtain the parameters values is presented in Fig.1 (a). In Figs.1 (b,c) we show schematic model of the bell indicating the position of the rotation axes of the bell the clapper and presenting parameters involved in the model. For simplicity, henceforth, we use term "bell" with respect to the combination of the bell and it's yoke which we treat as one solid element.

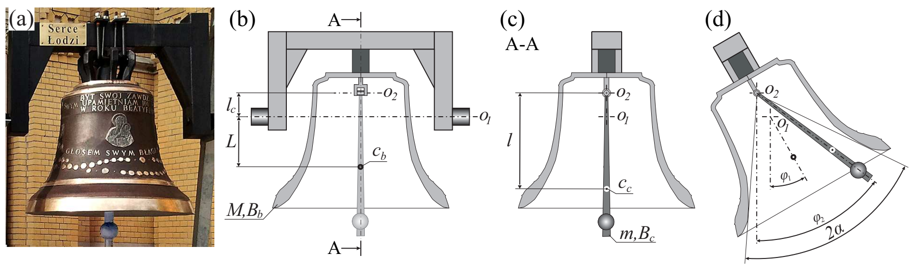

Fig. 1. *"The Heart of Lodz"* the biggest bell in the Cathedral Basilica of St Stanislaus Kostka (a) and it's schematic model (b,c,d) along with physical and geometrical quantities involved in the mathematical model of the system.

Parameter L describes the distance between the rotation axis of the bell and its center of gravity (point C_b), l is the distance between the rotation axis of the clapper and its center of gravity (point C_c). The distance between the bell's and the clapper's axes of rotation is given by parameter l_c. Mass of the bell is described by parameter M, while parameter B_b characterizes the bell's moment of inertia referred to it's axis of rotation. Similarly, parameter m describes the mass of the clapper and B_c stands for the clapper's moment of inertia referred to it's axis of rotation.

Considered model has two degrees of freedom. In Fig. 1 (d) we present two generalized coordinates that we use to describe the state of the system: the angle between the bell's axis and the downward vertical is given by φ_1 and the angle between the clapper's axis and downward vertical by φ_2. Parameter α (see 1 (d)) is used to describe the clapper to the bell impact condition which is as follows:

$$|\varphi_1 - \varphi_2| = \alpha \tag{1}$$

We use Lagrange equations of the second type and derive two coupled second order ODEs that describe the motion of the considered system (full derivation can be found in our previous publication [17]):

$$\begin{aligned} &\left(B_b + ml_c^2\right)\ddot{\varphi}_1 + ml_cl\ddot{\varphi}_2\cos\left(\varphi_2 - \varphi_1\right) - ml_cl\dot{\varphi}_2^2\sin\left(\varphi_2 - \varphi_1\right) \\ &+ \left(ML + ml_c\right)g\sin\varphi_1 + D_b\dot{\varphi}_1 - D_c\left(\dot{\varphi}_2 - \dot{\varphi}_1\right) = M_t(\varphi_1), \end{aligned} \tag{2}$$

$$\begin{aligned} &B_c\ddot{\varphi}_2 + ml_cl\ddot{\varphi}_1\cos\left(\varphi_2 - \varphi_1\right) + ml_cl\dot{\varphi}_1^2\sin\left(\varphi_2 - \varphi_1\right) \\ &+mgl\sin\varphi_2 + D_c\left(\dot{\varphi}_2 - \dot{\varphi}_1\right) = 0. \end{aligned} \tag{3}$$

where g stands for gravity and $M_t(\varphi_1)$ describes the effects of the linear motor propulsion. The motor is active - and excites the bell - when its deflection from vertical position is smaller than $\pi/15\,[rad]$ (12^o). The generalized momentum generated by the motor $M_t(\varphi_1)$ is given by the piecewise formula:

$$M_t(\varphi_1) = \begin{cases} T\,\mathrm{sgn}(\dot{\varphi}_1)\,\cos(7.5\varphi_1)\,, & if\ |\varphi_1| \leq \frac{\pi}{15} \\ \\ 0, & if\ |\varphi_1| > \frac{\pi}{15} \end{cases} \tag{4}$$

where T is the maximum achieved torque. Although the above expression is not an accurate description of the effects generated by the linear motor, in [17] we prove that it is able to reproduce the characteristics of the modern bells' propulsions.

There are eleven parameters involved in the mathematical model presented above. The parameters have the following values: $M = 2633\,[kg]$, $m = 57.4\,[kg]$, $B_b = 1375\,[kgm^2]$, $B_c = 45.15\,[kgm^2]$, $L = 0.236\,[m]$, $l = 0.739\,[m]$, $l_c = -0.1\,[m]$ and $\alpha = 30.65^o = 0.5349\,[rad]$, $D_c = 4.539\,[Nms]$, $D_b = 26.68[Nms]$, $T = 229.6[Nm]$. As aforementioned, all parameters values have been evaluated specifically for the purpose. For integration of the model described above we use the fourth-order Runge–Kutta method.

When the condition 1 is fulfilled we stop the integration process. Then, instead of analyzing the collision course, we restart simulation updating the initial conditions of equations 2 and 3 by switching the bell's and the clapper's angular velocities from the values before the impact to the ones after the impact. The angular velocities after the impact are obtained taking into account the energy dissipation and the conservation of the system's angular momentum that are expressed by the following formulas:

$$\frac{1}{2}B_c\left(\dot{\varphi}_{2,AI} - \dot{\varphi}_{1,AI}\right)^2 = k\frac{1}{2}B_c\left(\dot{\varphi}_{2,BI} - \dot{\varphi}_{1,BI}\right)^2, \tag{5}$$

$$\begin{aligned} &\left[B_b + ml_c^2 + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{1,BI} + \left[B_c + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{2,BI} = \\ &\left[B_b + ml_c^2 + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{1,AI} + \left[B_c + ml_c l\cos(\varphi_2 - \varphi_1)\right]\dot{\varphi}_{2,AI} \end{aligned} \tag{6}$$

where index AI stands for "after impact", index BI for "before impact" and parameter k is the coefficient of energy restitution. In our simulations we assume $k = 0.05$ referring to a series of experiments performed by Rupp et. al. [18]. In our previous investigation [17] we have analyzed influence of k on the response of the system and we have proved that system's dynamics barely changes for small alterations of parameter k ($\pm 20\%$). Hence, we claim that there is no need to further adjust the value of k for the considered bell. ODEs 2 and 3 together with the discreet model of impact create a hybrid dynamical system that can be used to simulate the behaviour of church bells.

2.1 Validation of the model

In this Section we investigate the full model and validate it focusing on normal ringing conditions. Our study is based on the bell in the Cathedral Basilica of

St Stanislaus Kostka and we check the reliability of the model by comparing the results of numerical simulations with experimentally obtained time traces of *"The Heart of Lodz"*.

We have launched the linear motor propulsion and after a start-up procedure, when the amplitude of the bell's motion stabilized, we have began recording with high-speed camera (Basler piA640-210gm). We have used black-and-white stickers to mark the position of the bell, the clapper and indicate reference length. We have used Kinovea software to create - basing on the recordings - the data-sheets with markers' abscissa and ordinate depending on time and processed the data in Mathematica software.

In Fig. 2 we compare the results of numerical simulations and time traces obtained from two separate recordings. In subplots (a,b) we present data collected from the first recording and in subplots (c,d) from the second one. Red lines in Fig. 2 represent periodic attractor obtained numerically and black dots correspond to experimental results. In subplots (a) and (c) we show time traces of the bell while subplots (b) and (d) are devoted to the clapper's behavior. Comparing the trajectories of the bell one can say that numerical results show remarkable agreement with the experiment. Simultaneously, time traces of the clapper do not show such a convergence as the clapper of the examined bell performed non-periodic motion. Divergence form numerically obtained periodic attractor is mainly visible around the moments of impact.

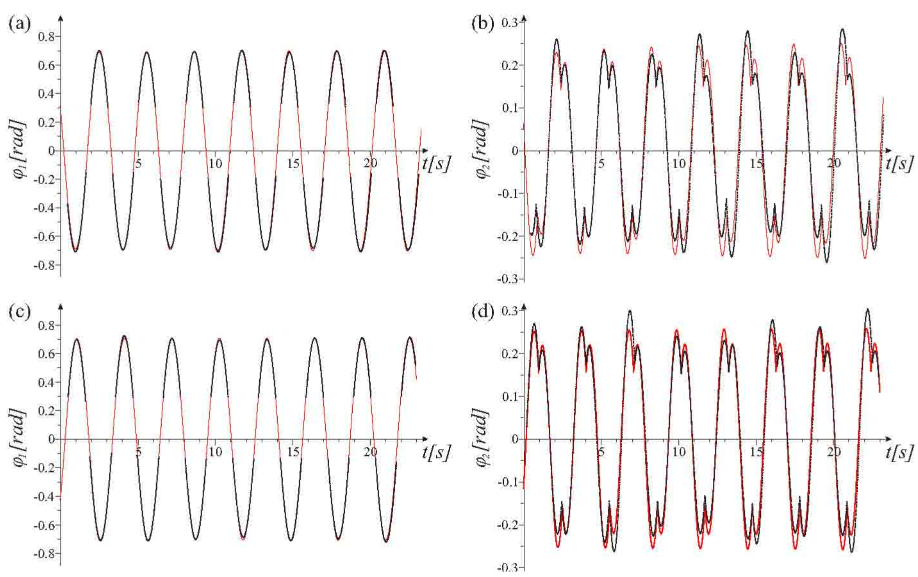

Fig. 2. Comparison between time traces of the bell (a,c) and the clapper (b,c) obtained numerically (red lines) and experimentally (black dots). Subplots (a,b) correspond to the data obtained from the first recording and (c,d) from the second one.

Analyzing Fig. 2 one can say that hybrid model introduced in this paper is able to simulate the bell's behavior with excellent accuracy and precisely determine crucial features of the clapper's motion such as: the period of motion, average amplitude of motion and predict moments of the clapper to the bell collisions. Still, slight divergence between the lines is visible around the moments of impacts. It is caused by the oscillations of the bell's shell which are not included in the mathematical model of the system. In [17] we describe this phenomenon in detail and prove that it do not compromise the reliability and practical significance of the considered model.

2.2 Influencing parameters

Most of the parameters involved in the model are self dependent. Moreover, we have to remember that we investigate a musical instrument, hence we cannot change some of its features that could affect the sound it generates. The two features that we can safely modify in real applications are the driving motor and the yoke of the bell. Therefore, we analyze how such changes influence the system's dynamics. As a reference, we use values of parameters characteristic for *"The Heart of Lodz"* and alter them to simulate modifications in the propulsion or mounting layout. We assume the linear motor driving that is described by piecewise function $M_t(\varphi_1)$ 4 and modify the output of the motor by changing the maximum generated torque T which we use as the first controlling parameter. To describe the modifications of the yoke we introduce the second parameter l_r whose meaning is described in Fig. 3.

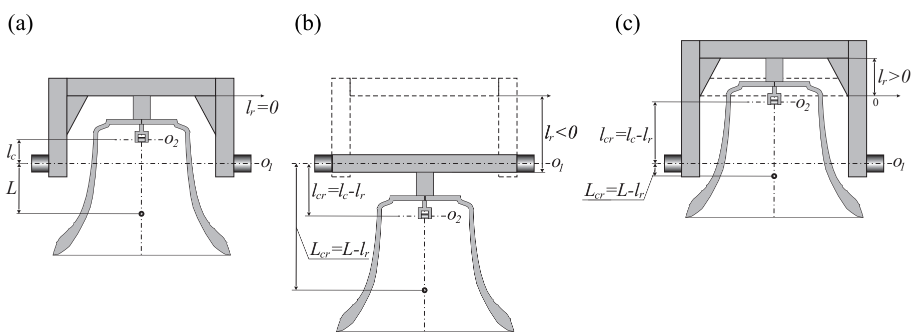

Fig. 3. Description of parameter l_r: (a) reference yoke design $l_r = 0$, (b) $l_r < 0$, (c) $l_r > 0$.

We take *"The Heart of Lodz"* as a reference yoke for which $l_r = 0$. If the bell's center of gravity is lowered, then $l_r < 0$ and as the value of l_r we take the distance by which the bell's center of gravity is shifted with respect to the reference yoke. Similarly, if we elevate the bell's center of gravity, we assume $l_r > 0$ and take its displacement as the value of l_r. As a maximum considered value of l_r we take $0.235\,[m]$ because for $l_r = 0.236\,[m]$ rotation axis of the bell goes through its center of mass.

It should be stressed out that each changes of l_r value affect other parameters of the model. So, when the value of l_r is changed the three different parameters should be swapped: L has to be replaced by L_r, l_c by l_{cr}, and finally B_b by B_{br}. New parameters L_r, l_{cr} and B_{br} are given by the following formulas:

$$
\begin{aligned}
L_r &= L - l_r \\
l_{cr} &= l_c - l_r \\
B_{br} &= \left(B_b - ML^2\right) + ML_r^2
\end{aligned}
\tag{7}
$$

3 Most common bells' working regimes

In this section we present and describe periodic attractors that can be considered as a proper working regimes and have practical applications. These regimes are often called ringing schemes and can be classified in groups that have common characteristics such as especially: number of collisions during one period of motion, course of collisions and time between them. In Fig. 4 we present 6 most common working regimes by showing phase portraits of the bell and the clapper (blue lines) and indicating the effects of collisions (red lines). Subplots of Fig. 4 were obtained for different values of parameters T and l_r.

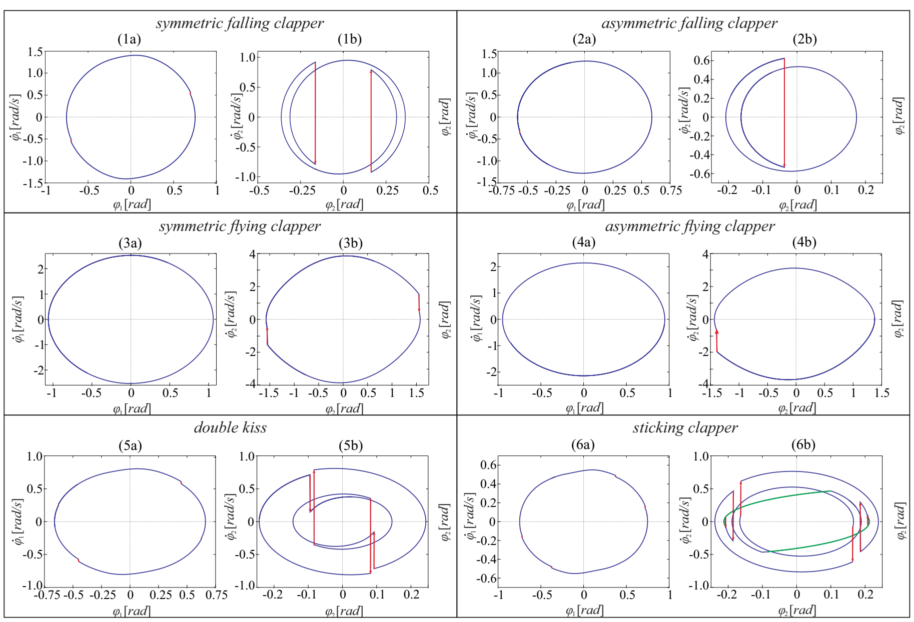

Fig. 4. Presentation of the most common working schemes of church bells. Subplots (a) and (b) present phase portraits of the bell and the clapper respectively.

We say that the bell works in a "falling clapper" manner if the collisions between the bell and the clapper occur when they perform an anti-phase motion (see Figs. 4 (1), (2)). This type of behavior is common for bells that are mounted in the European manner. In the "falling clapper" ringing scheme the amplitude of the clapper's motion is smaller than the bell's; and the clapper's velocity sign changes when collision occurs. We can distinguish a symmetric type of "falling clapper" with 2 collisions per one period of motion (Figs. 4 (1a-b) obtained for $T = 350\,[Nm]$ and $l_r = 0.05\,[m]$) and its asymmetric version with 1 impact per period (Figs. 4 (1a-b) obtained for $T = 150\,[Nm]$ and $l_r = -0.03\,[m]$). These ringing schemes differ mainly in the time intervals between the successive impacts.

The second characteristic working regime is called "flying clapper". In this regime collisions occur when the bell and the clapper perform in-phase motion. The amplitude of the clapper's motion is larger than the bell's; and the clapper's velocity sign remains the same after the collisions. The collisions have more gentle course than in the "falling clapper" manner, hence sometimes it may be difficult to achieve nice resounding of the bell. In Figs. 4 (3) and (4) we present two types of "flying clapper" behavior: symmetric attractor with 2 impacts per period obtained for $T = 450\,[Nm]$, $l_r = -0.91\,[m]$ and asymmetric one with only 1 impact per period that we receive for $T = 325\,[Nm]$ and $l_r = -1.21\,[m]$.

Bells mounted in English manner usually works in the "sticking clapper" regime in which the clapper and the bell remain in contact for a certain amount of time. In the considered system prior the sliding mode we observe a number of successive impacts (usually 3) that have a "falling clapper" course. In Fig. 4 (6) we show phase portraits of the bell (6a) and the clapper (6b) working in the "sticking clapper" manner obtained for $T = 125\,[Nm]$ and $l_r = 0.2\,[m]$. The energy amount that is transferred between the bell and the clapper decreases with each subsequent collision. Hence, the sound effects caused by each hit are different and not all collisions may be noticed by the listener.

Rarely we can observe the so-called "double kiss" working regime in which we observe 4 impacts per one period of motion (see Fig. 4 (5) obtained for $T = 175\,[Nm]$ and $l_r = 0.16\,[m]$). During one period the clapper hits each side of the bell's shell twice. The first collision on each side is in the "falling clapper" manner while the second impact has "flying clapper" course. This behavior is especially attractive for the listeners but it is difficult to achieve.

Apart form the described above we can also observe stable periodic attractors with no collisions, but then no sound is produced and the bell can not work as a musical instrument. Unfortunately, no impacting attractors occur in wide range of T and l_r values. Moreover, we can distinguish other periodic attractors that can be successfully employed such as asymmetric "flying clapper" behavior with doubled period and 4 impacts per period that can be easily taken as a typical "flying clapper". Similarly, a quasi-periodic attractor with almost equal time intervals between the subsequent impacts can sound almost like a periodic ringing scheme. In the next Section we analyze how the most common working schemes can be achieved by proper designing the propulsion mechanism and the yoke of the bell.

4 Influence of the yoke design and forcing amplitude on the system's dynamics

The yoke's design - described by parameter l_r - and the amplitude of forcing - parameter T - define the working regime of the bell. In this section we analyze how these parameters influence the response of the system. In Fig. 5 we show the result of series of numerical simulations obtained using 154 different sets of parameters values from the following ranges: $l_r \epsilon \langle -1.3, 0.23 \rangle$ $[m]$ and $T \epsilon \langle 100, 625 \rangle$ $[Nm]$. For each set of parameters we start the simulation from zero initial conditions ($\varphi_1 = 0$, $\varphi_2 = 0$, $\dot{\varphi}_1 = 0$, $\dot{\varphi}_2 = 0$) and mark on the plot which ringing scheme we obtain. We concern only the solutions that basins of attraction contain zero initial conditions ($\varphi_1 = 0$, $\varphi_2 = 0$, $\dot{\varphi}_1 = 0$, $\dot{\varphi}_2 = 0$). Such approach is practically justified because in most cases the bell and the clapper start their motion from hanging down position with zero velocities.

We consider 7 most characteristic types of the bells behavior that are described in detail in the previous Section.

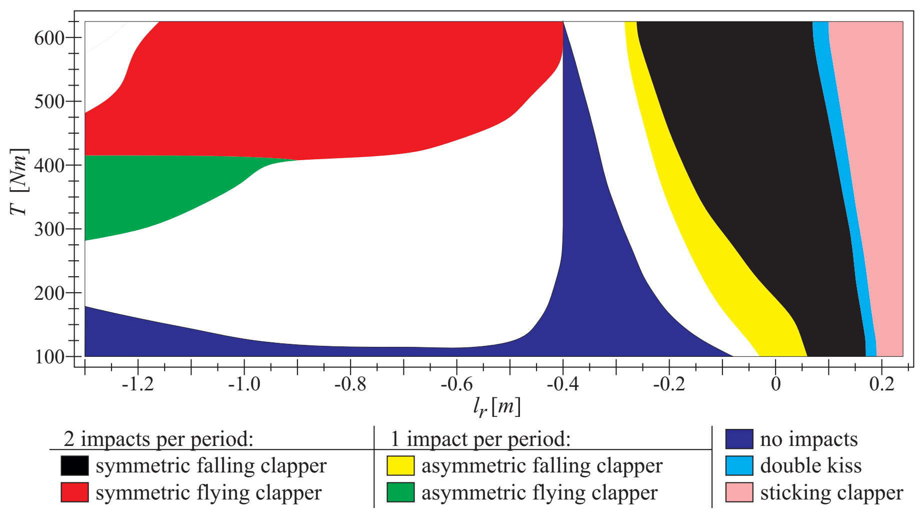

Fig. 5. Two parameter ringing schemes diagram showing the behavior of the system for different values of l_r and T.

Analyzing Fig. 5 we see that the biggest areas correspond to the three most common ringing schemes: symmetric "falling clapper" (black) and "flying clapper" (red) and "sticking clapper" (pink) that is typical for the bells mounted in the English manner. Thanks to that, these behaviors are relatively easy to achieve and remain even after long period of time when values of some parameters can change a bit (for example maximum torque generated by the linear motor). We see that, in general, the design of the yoke determines how the bell will operate. For $l_r < -0.4\,[m]$ we can observe "flying clapper" with 2 (red) or 1 (green) impacts per period but these ringing schemes can be achieved only when T is bigger than some threshold value which decreases with the decrease of l_r. If T is not sufficient we will not observe any impacts (blue) or the system will reach different attractor (white) - periodic or non-periodic one. For $l_r \epsilon \, (-0.4. - 0.284)$ $[m]$ we cannot obtain any of the analyzed ringing schemes despite the forcing amplitude. Hence, when the yoke is designed improperly it may be impossible to force the system to ensure proper operation. For $l_r > -0.284 \, [m]$ the system can work in the following manner: "falling clapper" with 1 (yellow) or 2 (black) impacts per period, "double kiss" (light blue) or "sticking clapper" (pink). Each of these ringing schemes can be achieved despite the value of T but the yoke should be designed for the purpose. Hence, for these working regimes there is no need to use very powerful driving motors and the system can work more efficiently. In our previous publication [19] we consider transitions between different dynamicals states of the yoke-bell-clapper system and analyse the time that is needed to reach presumed solution.

5 Conclusions

In this paper we investigate a plethora of different dynamical behaviors encountered during the analysis of the hybrid dynamical model of the church bell. The model is developed basing on the bell in the Cathedral Basilica of Stanislaus Kosta, Lodz, Poland. Its parameters values are determined during the series of measurements and experiments involving the bell [17]. Finally, we validate the model by comparing the results of numerical simulations with experimental data and prove that it provides all crucial information about the system's dynamics which is beneficial for bell-founders, bell-hangers and engineers working on bells or bell towers.

In the next part of the paper we focus on solutions that can be considered as a proper working regimes of the instrument. We present and characterize the 6 most common behaviours and present a method that can be used to determine the conditions under which given type of behavior can be achieved. We use the amplitude of the forcing T and the yoke design (described by the parameter l_r) as the influencing parameters and develop two parameters ringing scheme diagrams. Such plots provide full information on how the geometry of the yoke and maximum output of the driving motor influence the dynamics of the system.

Ringing scheme charts can be calculated for any bell and used to design its yoke and propulsion. Presented tools can help to improve working conditions of existing bells as well as during the design of mounting layouts for new instruments. Thus, using the numerical analysis of the systems dynamics we can ensure that the bell will work properly and reliably for ages and regardless of small changes of parameters.

Acknowledgment

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Linearization of an invertible bounded iteration in $\mathbb{R}^d$

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Abstract. In this paper, we study an iteration f in $\mathbb{R}^d$ defined by a diffeomorphism polynomial bounded. So, the image of the invariant curves is relatively compact and we can use the Fourier-Bohr's representation in the set of almost periodic function ($A.P.$). These curves have asymptotically a parameterization with Weierstrass-Mandelbrot's functions depending on fluctuation's parameters. So, self-similarity and fractal dimension calculus are justified. We apply these results to partial differential calculus.

Key words: Bounded invertible iterations, invariant curves, fluctuation's parameters, Weierstrass-Mendelbrot functions, Hénon. Navier-stokes.

Lead Paragraph

A clear definition of the chaos is a very difficult thing. Mathematically speaking, a chaos is an iteration with a hieratic behaviour, but nobody has a simple mathematical definition. Nevertheless, computing simulations give easily numerous examples. One of the best definition of this situation is the highly sensibility to initial conditions. But this negative definition yields to an impossible long-term prediction. So, each author uses a lot of hypothesis to describe his particular context rendering quite impossible an unified vision of the methods and results. We use here the very restrictive hypothesis of a bounded bijective polynomial iteration in $\mathbb{R}^d$. Under this assumption, we obtain a good and general description of the behaviour in terms of series of Weierstrass-Mandelbrod's functions. This approach justifies some methods used by researchers and gives the exact conditions to use it. Another interesting aspect of this method is the possible various generalizations to study more complicate situations.

1. Introduction

A. Definitions and hypothesis

Let f be an application de $\mathbb{R}^d$ in $\mathbb{R}^d$. We call iteration the same application f when it is iterated indefinitely: $f^{(p)}$ with $f^{(p)} = f \circ f^{(p-1)}$. We have yet study this problem with probabilistic methods such as the Perron-Frobenius operator$[5]$. But here, f is a ***polynomial diffeomorphism and applies a bounded set*** $C \subset \mathbb{R}^d$ ***in itself***: $f(C) \subset C$. We shall see that the diffeomorphism's hypothesis leads to a deterministic approach.

We recall the definitions of some sets invariant under iterations and H0 hypothesis.

1. Definitions of invariant points or cycles

Let $f(\boldsymbol{0}) = \boldsymbol{0}$ be a fixed point of f well isolated in C and λ the eigen values of the linear part of f at $\boldsymbol{0}$. Iteration of f induces other invariant points under $f^{(p)}$. These points form cycles such as $f^{(p)}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$ with $f(\boldsymbol{\alpha}) \neq \boldsymbol{\alpha}$. Let $Fix(f^{(p)})$ this set and $f_\bullet$ the collection of iterations $f_\bullet = (f, f^{(2)}, ..., f^{(p)} ...)$. All that we say about some fixed point $f(\boldsymbol{0}) = \boldsymbol{0}$ will be thru for all the collection $Fix(f_\bullet) = \cup_p Fix(f^{(p)}) \cap C$.

2. Hypothesis H0

f is a bounded polynomial diffeomorphism in C. *The eigen values* λ *of* $f'(\boldsymbol{0})$ *at* $\boldsymbol{0}$ *are* ***not resonant:*** $1 \neq \lambda^{n} \middle| \forall n \in \mathbb{Z}^d$. *All the* $q < d$ *eigen values* $|\lambda| > 1$ *are* ***positive transcendental***.

If some $\lambda < -1$ are negative, by iterating f twice times, we come back in the positive case: $\boldsymbol{a}(\lambda^2 \boldsymbol{t}) = f \circ f(\boldsymbol{a}(\boldsymbol{t}))$. So, **we always work with all** $\lambda > 1 \in \mathbb{R}^{+q}$.

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B. Recall of known results for a C^k - diffeomorphism f in $\mathbb{R}^d$, $k \geq 1$

We consider new invariant sets in our case of a C^k - diffeomorphism f in $\mathbb{R}^d$, $k \geq 1$: ***linearization functions*** and ***invariant functions.*** These functions, quite reciprocal each-others, define **invariant curves** and are well known. They were studied by Steinberg $[9]$. With Steinberg's series, in the neighbourhood of $\boldsymbol{0}$, we can define d ***linearization*** C^k ***-functions*** $\varphi_\ell \circ f(\boldsymbol{a}) = \lambda_\ell \varphi_\ell(\boldsymbol{a}),\ \ell = 1,..,d$, under conditions of no resonance. Under the same conditions, we admit that we solve the functional equation $\boldsymbol{a}(\lambda \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})), \boldsymbol{t} \in \mathbb{R}^d$ with the Steinberg's series. Then, we get d ***invariant functions*** $\boldsymbol{a}(\boldsymbol{t})$: $a_\ell(\lambda \boldsymbol{t}) = f_\ell(\boldsymbol{a}(\boldsymbol{t})\ \ell = 1,..,d$. More, φ and $\boldsymbol{a}$ are unique for a unit Jacobian matrix at $\boldsymbol{0}$.

1. The linearization functions $\varphi_\ell(\boldsymbol{a})$

If $|\lambda_\ell| < 1$, then we obtain asymptotically $\varphi_\ell(a) = 0$ by iteration of f. We have a manifold defined by these equations. These results are not very useful here.

2. Invariant functions $\boldsymbol{a}(\boldsymbol{t})$ and the fluctuation's parameter $\boldsymbol{t}$

Here, we pay a special attention to $\boldsymbol{a}(\boldsymbol{t})$ constructed with Steinberg's series defined in the neighbourhood of $\boldsymbol{0}$. Let $\rho > 0$ be the radius of convergence of $\boldsymbol{a}(\boldsymbol{t})$. Then, taking $\boldsymbol{t} : |\boldsymbol{t}| \leq \rho_0 < \rho$, we extend $\boldsymbol{a}(\lambda^p \boldsymbol{t}) \in C$ for $\forall p \in Z$ with sequential iterations for $\forall \boldsymbol{t} \in \mathbb{R}^{+d}$. This unique solution will be useful. When we iterate of $f^{(p)}$ with $p \to \infty$, as the coordinates of $\boldsymbol{t}$ for $|\lambda| < 1$ tend to 0, we are only interested by the q eigen values $\lambda \in \mathbb{R}^{+q} \| \lambda| > 1$ and the corresponding $\boldsymbol{t} \in \mathbb{R}^{+q}$. $\boldsymbol{t}$ is called **fluctuation's parameter.**

3. Remarks

If we start from an arbitrary point $\boldsymbol{a}_0 \in C$ and we iterate indefinitely this point, the set obtained, called sometimes orbit, doesn't verify E except if $\boldsymbol{a}_0 = \boldsymbol{a}(\boldsymbol{t}_0, \boldsymbol{t}'_0)$. In this case, iteration verifies asymptotically equation E by continuity. But, we have for each point of $Fix(f_\bullet)$ curves and invariant measures: we have a « *mille-feuilles* » of solutions with a well-known very hard complexity. So, if uniqueness of the fixed point $\boldsymbol{0}$ in C is not sure, or if it is not well isolated, we can meet many difficulties. Solutions found here are local.

C. The problem: find the solutions of the equation E : $\boldsymbol{a}(\lambda \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})) | \lambda > 1, \boldsymbol{t} \in \mathbb{R}^{+q}$.

The mean idea of this paper is the following: Under H0, the adherence of the iteration and the invariant function $\boldsymbol{a}(\boldsymbol{t})$ is compact. So, the image of $\boldsymbol{a}(\boldsymbol{t})$ is relatively compact. Then, we can apply the Bohr's results about the almost periodic functions and represent $\boldsymbol{a}(\boldsymbol{t})$ in an almost periodic series. But, equation E implies many conditions on the almost period (a.p;) and on the Fourier's coefficients. Then, we can solve in general the problem in terms of Weierstrass-Mandelbrot functions and obtain a solution. But, as we have say about $Fix(f_\bullet)$, if start from an arbitrary point in C, it is very difficult to describe the convergence to one or to an another set of $Fix(f_\bullet)$.

Here, we search only the unique solution of E (invariant curves corresponding to $\lambda > 1$) in the set of the almost periodic functions (A.P.). We shall see that the solution is also auto similar (A.S.).

2. Bohr's almost periodic functions in a Banach [1] with application to iterations

Let A.P. be the set of the almost periodic functions bounded in $\mathbb{R}^d$. Let $\boldsymbol{a}(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^{+q}, q \leq d$.

A. Bochner's definition (recall $[2]$,$[3]$)

A function $\boldsymbol{a}(\boldsymbol{t}) \in A.P.$ in the Bohr's sense, if and only if, the set of the translated functions by all $\boldsymbol{c}$ $\left\{\boldsymbol{a}(\boldsymbol{t}+\boldsymbol{c}) | \boldsymbol{c}, \boldsymbol{t} \in \mathbb{R}^{+q}\right\}$ is relatively compact for the uniform convergence's topology.

Now, we have an arsenal of results near of the Fourier's series$[1]$. The following theorem 1 is proved in Banach's spaces. Here we use it only in $\mathbb{R}^d$.

B. Theorem 1 (recall of the Bohr's approximation $[1]$)

Let a trigonometric function: $P_n(\boldsymbol{t}) = \Sigma_{k=1}^{n} \boldsymbol{c}_k e^{i\mu_k \boldsymbol{t}} \in \mathbb{R}^d$ *where* $\mu_k, \boldsymbol{t} \in \mathbb{R}^q$ $\boldsymbol{c}_k \in \mathbb{R}^d$ *and* $\mu_k \boldsymbol{t}$ *is scalar product. If* $\boldsymbol{a}(\boldsymbol{t}) \in A.P.$*, we have a sequence of* $\boldsymbol{P}_n(\boldsymbol{t})$ *such as for* $\forall \varepsilon > 0$ *:*

$$\sup_{\boldsymbol{t} \in \mathbb{R}^q} \|\boldsymbol{a}(\boldsymbol{t}) - \boldsymbol{P}_n(\boldsymbol{t})\| < \varepsilon .$$

If $\boldsymbol{c}(\mu) = \boldsymbol{F}_\mu(\boldsymbol{a}) = \lim_{T \to \infty} \frac{1}{(2T)^q} \int .. \int_{-T}^{T} \boldsymbol{a}(\boldsymbol{t}) e^{-i\mu t} \, d\boldsymbol{t}$ *is the vector of Fourier-Bohr's coefficients,*

then $\Lambda(a) = \{\mu \in \mathbb{R}, c(\mu) \neq 0\}$ *is the set of the almost period (a.p.);* Λ *is countable.* $\boldsymbol{a}(\boldsymbol{t})$ *is uniformly continuous with Fourier's series:* $\boldsymbol{a}(\boldsymbol{t}) \sim \sum_{\mu \in \Lambda} \boldsymbol{c}(\mu) e^{i\mu t}$*. This representation is unique.*

We have the mean-square convergence and the Perceval's equality.

It is the same to study relative compactness either with $(\boldsymbol{t} + \boldsymbol{c}), \boldsymbol{t}$, or $(\boldsymbol{tc})$ where $\boldsymbol{c}, \boldsymbol{t} \in \mathbb{R}^{+q}$ because we can write $\boldsymbol{t} + \boldsymbol{c} = \boldsymbol{c'}\boldsymbol{t}, \forall \boldsymbol{t} \neq \boldsymbol{0}$.

C. Theorem 2

Under H0, $\{\boldsymbol{a}(\boldsymbol{t}), \boldsymbol{t} \in \mathbb{R}^{+q}\}$ *and* $\boldsymbol{a}(\boldsymbol{ct})$ *for all* $\boldsymbol{c}$ *are relatively compact and* $\boldsymbol{a}(\boldsymbol{t}) \in A.P.$ *Let* $\boldsymbol{t} = \boldsymbol{g}(\boldsymbol{u})$ *a* C^0 *diffeomorphism of* $\mathbb{R}^q$ *in* $\mathbb{R}^q$. $\boldsymbol{a} \circ \boldsymbol{g}(\boldsymbol{u})$ *has again an almost periodic parameterization.*

Let $A_N = \{(\boldsymbol{a}(\lambda^p \boldsymbol{t}); \|\boldsymbol{t}\| \leq \rho_0, p \leq N\}$. Its closure $\widehat{A}$ is closed and contained in C bounded, so is compact. The image of $\boldsymbol{a}(\boldsymbol{t})$ is relatively compact. We « translate» $\boldsymbol{t}$ with $\boldsymbol{c}$: $\boldsymbol{ct}$ such as $\boldsymbol{a}(\boldsymbol{ct}) \in C$. Let n be sufficiently large in order to have $\boldsymbol{c} = \alpha \lambda^n$ with $\|\alpha\| < 1$ and $\|\alpha \boldsymbol{t}\| < \|\boldsymbol{t}\| < \rho_0$, $\boldsymbol{ct} = \alpha \boldsymbol{t} \lambda^n$. As f is invertible, $f^{-(n)} \boldsymbol{a}(\boldsymbol{ct}) = \boldsymbol{a}(\alpha \boldsymbol{t})$ with $|\alpha \boldsymbol{t}| \leq \rho_0$ such as A_N has adherence $\widehat{A}$. So $\boldsymbol{a}(\boldsymbol{t}) \in A.P.$. If $\boldsymbol{t} = \boldsymbol{g}(\boldsymbol{u})$ a C^0 diffeomorphism of $\mathbb{R}^q$ in $\mathbb{R}^q$, we have again an almost periodic parameterization.

Example: $\boldsymbol{g} = (\exp(\boldsymbol{u}_1, ..., \exp(\boldsymbol{u}_q))$ defines the transformation $\boldsymbol{u} = \log \boldsymbol{t}$ and $\alpha = \log \lambda$ where $\boldsymbol{t}, \lambda \in \mathbb{R}^{+q}$. We can write $\boldsymbol{a}(\lambda \boldsymbol{t}) = \boldsymbol{a} \circ \exp(\log \lambda \boldsymbol{t})) = \boldsymbol{a} \circ \exp(\alpha + \boldsymbol{u})$ and we have an almost periodic parameterization.

3. Solution of the equation E: $\boldsymbol{a}(\lambda \boldsymbol{t}) = f(\boldsymbol{a}(\boldsymbol{t})), \boldsymbol{t} \in \mathbb{R}^{+q}$ in $A.P.$

As $\boldsymbol{a}(\boldsymbol{t}) \in A.P.$, we search an asymptotically auto similar solution.

Definition of an auto similar function

A function $\boldsymbol{w}(\boldsymbol{t})$ *is auto similar if:* $\boldsymbol{w}(\lambda \boldsymbol{t}) = r^{-1} \boldsymbol{w}(\boldsymbol{t})$ *where* $r < 1$ *for* $\lambda > 1$ *and* $\boldsymbol{t} \in \mathbb{R}^{+q}$.

The Weierstrass-Mandelbrot (W-M) function $[8]$ $w_\lambda(t) = \Sigma_{k \in \mathbb{Z}} r^k (1 - e^{i\lambda^k \boldsymbol{t}})$ is auto similar where $r = \lambda^{D-2} < 1$ is the ratio and $\lambda > 1$ the almost period (a.p.) of w and $t \in \mathbb{R}^+$.

A. Theorem of linearization 3

Under H0, the a.p. μ *solutions of E are* $\mu = m\lambda^k \mid k \in \mathbb{Z}, m \in \mathbb{N}$ *and* $\boldsymbol{a}(\boldsymbol{t})$ *is written:*

$$\boldsymbol{a}(\boldsymbol{t}) = \Sigma_{m \in \mathbb{N}} \boldsymbol{a}_m(\boldsymbol{t})$$

with:

$$\boldsymbol{a}_m(\boldsymbol{t}) = \Sigma_{k \in \mathbb{Z}} \boldsymbol{c}(m\lambda^k)(1 - e^{im\lambda^k \boldsymbol{t}})$$

If $m = 1$, $\boldsymbol{a}_1(\boldsymbol{t}) = \Sigma_{k \in \mathbb{Z}} \boldsymbol{c}(\lambda^k)(1 - e^{i\lambda^k \boldsymbol{t}})$ *is a series of W-M functions q- dimensional:*

$$\boldsymbol{a}_1(\boldsymbol{t}) = \Sigma_{\ell=1}^{\ell=q} \boldsymbol{c}_\ell w_{\lambda_\ell}(t_\ell)$$

Where:

- $\boldsymbol{c}_\ell$ *is eigen vector of* $f'(\boldsymbol{0})\boldsymbol{c} = \lambda \boldsymbol{c}$ *for the eigen value* $\lambda_\ell > 1$,

- $w_\ell(t)$, *is the one-dimensional* W-M function*:* $w_\ell(t)=\Sigma_k r_\ell^k(1-e^{i\lambda_\ell^k t_\ell})$ *with* $r_\ell^{-1}=\lambda_\ell^2$.
But, we denote that $\boldsymbol{a}_1(\boldsymbol{t})$ is not a ***solution of E.***

The Taylor's formula of f in dimension d is written: $f(\boldsymbol{a})=\Sigma_{n=0}^{n=n_0}f^n(\boldsymbol{0})\boldsymbol{a}^n/n!$.
Replacing $\boldsymbol{a}$ with $\boldsymbol{a}(\boldsymbol{t})\in A.P.$ in the monomials, the equation E implies:
* *for the exponents and the almost period a*.p.;

$\Sigma_{j\mu'}(m_j\mu'-\lambda\mu)=0$ where the integers $\Sigma_j m_j\le n_0$ (n_0 degree of f) and (μ',μ) are a.p. If $\mu\in\Lambda(a)$ is an a.p. of $\boldsymbol{a}(\boldsymbol{t})$, on the one hand, $\lambda\mu\in\Lambda(a)$ is an a.p., on the other hand, $m_j\mu\in\Lambda(a)$. As f is invertible, all the a.p. are: $\mu'=\mu\lambda^k\in\Lambda(a)|\ k\in\mathbb{Z}$. multiple integers of $m\mu\in\Lambda(a)|m\in\mathbb{N}$. Then, all the a.p. are:

$$\mu=m\lambda^k=\Sigma_j m_j\lambda^{kj}\in\Lambda(a)|\ k_j\in\mathbb{Z}$$

But, as f is polynomial of n_0 degree, the equality on the a.p. contains less than n_0+1 terms and a finite condition **C** can be written:

$$\Sigma_j m_j\lambda^j-m\lambda^k=0\,.$$

Then, as $\boldsymbol{a}(\boldsymbol{t})$ is almost periodic and continue at the origin, we can write:

$$\boldsymbol{a}(\boldsymbol{t})=\Sigma_{m\in\mathbb{N}}(\Sigma_{k\in\mathbb{Z}}\boldsymbol{c}(m\lambda^k)(1-e^{im\lambda^k\boldsymbol{t}}))$$

We observe also that all the integer exponents must be integer multiples of the exponents of the polynomials f.

* *for the Fourier's coefficients*:

The Fourier's coefficients of $\boldsymbol{a}(\boldsymbol{t})$ are obtained Fourier's transformation of $\boldsymbol{a}(\boldsymbol{t})$ for each a.p. $m\lambda^k$. So, we apply the Fourier's transformation $\boldsymbol{F}_{m\lambda^k}$ to the equation **E**:

$$\boldsymbol{c}(m\lambda^{k-1})/|\lambda|=\boldsymbol{F}_{m\lambda^k}(\boldsymbol{a}(\boldsymbol{t}))\ \text{with: } |\lambda|=\Pi_{j=1}^{j=q}\lambda_j$$

We can see that $\boldsymbol{F}_{m\lambda^k}(\boldsymbol{a}(\boldsymbol{t}))$ is not easy to compute. We order the computation according to the increasing $m\in\mathbb{N}$.

* *First, we consider* $m=1$.

The theorem 4 will extend the computation to all m in particular situations. If $m=1$, condition **C** becomes: $\Sigma_j m_j\lambda^j-\lambda^k=0$. If the eigen values $|\lambda|>1$ are real positive transcendental, all the coefficients of **C** must be null: $m_j=0|j\ne k$ except $m_k=1$ As f is polynomial, we study only the transformation $\boldsymbol{F}_k\left(f^{n+m}(\boldsymbol{0})(a_i(t))^n(a_j(t)^m/n!m!\right)$ (for the a.p. λ^k) of a monomial of 2 arbitrary coordinates of $\boldsymbol{a}(t)$: $a_i(t)$ and $a_j(t)$. We note: $\mu_i=\lambda^{s_i}$ .The multinomial formula gives:

$$(a_i(t))^n=n!\Sigma_{r=1}^d\Sigma_\mu\Pi_{p_r=1}^n(c_i(\mu_i)^{p_r}e^{i(\Sigma p_r\mu_i)t}/p_r!$$
$$(a_j(t)^m=m!\Sigma_{r=1}^d\Sigma_{\mu'}\Pi_{q_s=0}^n(c_j(\mu_j)^{q_s}e^{i(\Sigma q_s\mu_j)t}/q_s!\,.$$

The general term GT of the product is:

$$TG(a_i(t))^n(a_j(t))^m=\Sigma\Pi_{p_r=1}^n c^{te}(c_i(\mu_i)^{p_r}\Pi_{q_s=0}^n(c_j(\mu_j)^{q_s}e^{i(\Sigma p_r\mu_i+\Sigma q_s\mu_j)t}\,.$$

So, all monomials of degree two or more are null except if : $\Sigma p_r\mu_i+\Sigma q_s\mu_j=\mu_k/\lambda$. Then: $i=\ell\ or\ j=\ell$ with $\mu_i=\mu_k/\lambda\ \ or\ \ \mu_j=\mu_k/\lambda$. All the other coefficients are null. So:

$$\boldsymbol{F}_k\left(f^{n+m}(\boldsymbol{0})(a_i(t))^n(a_j(t)^m/n!m!\right)=\Sigma_{\ell=i}^j f'(0)c_\ell(\mu_k)\,.$$

If $m=1$ and if we note $\boldsymbol{a}_1(\boldsymbol{t})$ the solution of E in this case, the previous result means:

$$\boldsymbol{c}(\lambda^{k-1})/|\lambda| = f'(\boldsymbol{0})\boldsymbol{c}(\lambda^k)$$

And, $\boldsymbol{c}(\lambda^k)$ is eigen vector of $f'(\boldsymbol{0})$ for the eigen values $\lambda > 1$. We can write for an eigen value λ_ℓ and an eigen vector $\boldsymbol{c}_\ell$:

$$\boldsymbol{c}_\ell(m\lambda^k) = \alpha_\ell(m\lambda^k)\boldsymbol{c}_\ell .$$

Then: $$\alpha_\ell(m\lambda^{k-1})/|\lambda| = \lambda_\ell \alpha_\ell(m\lambda^k)$$

The solution of this recurrence is: $\alpha_\ell(\lambda^k) = \alpha_\ell r_\ell^k$ with $r_\ell^{-1} = |\lambda|\lambda_\ell$.

Then, a coordinate $\boldsymbol{a}_{1\ell}(\boldsymbol{t})$ of $\boldsymbol{a}_1(\boldsymbol{t})$ is: $\boldsymbol{a}_{1\ell}(\boldsymbol{t}) = \boldsymbol{c}_\ell(\Sigma_{k\in\mathbb{Z}}\alpha_\ell(\lambda^k)(1-e^{i\lambda^k \boldsymbol{t}}))$

$$\boldsymbol{a}_{1\ell}(\boldsymbol{t}) = \boldsymbol{c}_\ell \alpha_\ell w_{\lambda_\ell}(\boldsymbol{t})$$

with $w_{\lambda_\ell}(\boldsymbol{t}) = \Sigma_{k\in\mathbb{Z}} r_\ell^k (1-e^{i\Sigma_{j=1}^{j=q}\lambda_j^k \boldsymbol{t}_j})$

We must verify the continuity of $\boldsymbol{a}_1(\boldsymbol{t})$ at the origin: if $t_j = 0 \mid j \neq \ell$, we observe that $\boldsymbol{F}_{\lambda^k}(\boldsymbol{a}(\lambda\boldsymbol{t}))$ is now: $$\boldsymbol{F}_{\lambda^k}(\boldsymbol{a}(\lambda\boldsymbol{t})) = \boldsymbol{c}(m\lambda^{k-1})|\lambda_j|/|\lambda| .$$

To keep the continuity when $t_j \to 0$, $\Sigma_{j=1}^{j=q}\lambda_j^k \boldsymbol{t}_j$ must be reduced to $\lambda_\ell^k \boldsymbol{t}_\ell$ and $r_\ell^{-1} = |\lambda_\ell|\lambda_\ell = \lambda_\ell^2$.

So, the Fourier's transform is reduced to the t_ℓ alone: $w_{\lambda_\ell}(\boldsymbol{t}) = w_{\lambda_\ell}(t_\ell)$.

$$\boldsymbol{a}_1(\boldsymbol{t}) = \Sigma_\ell \boldsymbol{c}_\ell w_{\lambda_\ell}(t_\ell)$$

We denote that $\boldsymbol{a}_1(\boldsymbol{t})$ is not a **solution of E:** If $\boldsymbol{a}_1(\boldsymbol{t})$ is solution of E, we have: $\boldsymbol{a}_1(\lambda\boldsymbol{t}) = r^{-1}\boldsymbol{a}_1(\boldsymbol{t}) = f(\boldsymbol{a}_1(\boldsymbol{t}))$ with $r<1$. By iteration, we have: $\boldsymbol{a}_1(\boldsymbol{t}) = \boldsymbol{0}$.

Remarks:

- If λ is algebraic, the solution is not linear. This case, of null measure, is more difficult because the relations on the coefficients are not linear.

- When we have many eigen values λ greater than the unit, so many $r<1$, we must take $r_0 = \min r_\ell$, express all $r_\ell = r_0^\alpha \mid \alpha < 1$, then the process of identification used is quite the same as in the following theorem, but more complicate to present.

We have found a solution $\boldsymbol{a}_1(\boldsymbol{t}) \in A.S.A.P.$. Now, we construct the general solution $\boldsymbol{a}(\boldsymbol{t}) \in A.S.A.P.$ verifying E: We present here the process of identification when **we have only one eigen value $\lambda > 1$, and one ratio $r<1$.** We note $\varepsilon = r^p$ and $\boldsymbol{a}_m(\lambda^{-p}\boldsymbol{t}) = r^{pm}\boldsymbol{a}_m(\boldsymbol{t}) = \varepsilon^m \boldsymbol{a}_m(\boldsymbol{t})$. So, we write a priori $\boldsymbol{a}(\lambda^{-p}\boldsymbol{t}) = \Sigma\varepsilon^k \boldsymbol{a}_k(\boldsymbol{t})$. Let $g_m(f, A(\boldsymbol{t})) = \Sigma_{n=2}^{n=m-1} f^n(\boldsymbol{0})\boldsymbol{A}_{n,m}(\boldsymbol{t})$ and $\boldsymbol{A}_{n,m}(\boldsymbol{t})$ is polynomial in $(\boldsymbol{a}_1(\boldsymbol{t})\boldsymbol{a}_2(\boldsymbol{t})...\boldsymbol{a}_{m-1}(\boldsymbol{t}))$ such as $\boldsymbol{A}_{n,m}(\lambda\boldsymbol{t}) = \varepsilon^m \boldsymbol{A}_{n,m}(\boldsymbol{t})$. We proceed by identification gradually. If the process converges, the gap becomes asymptotically null.

B. Theorem of recurrence 4

Under H0, if we have only one eigen value $\lambda > 1$, the unique solution of E: $\boldsymbol{a}(\boldsymbol{t}) = \Sigma \boldsymbol{a}_m(\boldsymbol{t}) \in A.S.A.P.$ is obtained with a finite sequence of linear relations with $\boldsymbol{a}_1(\boldsymbol{t})$:

$(r^{-1} - f'(0)\boldsymbol{a}_m(\boldsymbol{t}) = g_m(f, A(\boldsymbol{t})) + f^m(\boldsymbol{0})\boldsymbol{a}_1(\boldsymbol{t})^m / m!$ *if* $2 \le m \le n_0$,

$(r^{-1} - f'(0)\boldsymbol{a}_m(\boldsymbol{t}) = g_m(f, A(\boldsymbol{t}))$ *if* $n_0 + 1 \le m \le n_0^2$,

We take advantage of the function $\boldsymbol{a}_1(\boldsymbol{t})$, by observing that the invariant functions are asymptotically tangential to the eigen vectors at $\boldsymbol{0}$ and $\boldsymbol{a}_\infty(\boldsymbol{t})$ are these asymptotes.

With $\varepsilon = r^p$, equation E is:

$$\boldsymbol{a}(\lambda^{-p+1}\boldsymbol{t}) = f(\boldsymbol{a}(\lambda^{-p}\boldsymbol{t}) = r^{-1}(\Sigma_{m=1}^{m=N}\varepsilon^m \boldsymbol{a}_m(\boldsymbol{t})) = f(\Sigma_{m=1}^{m=N}\varepsilon^m \boldsymbol{a}_m(\boldsymbol{t})) .$$

We order $f(\Sigma\varepsilon^m \boldsymbol{a}_m(\boldsymbol{t}))$ according to the power of $\varepsilon : f(\Sigma_{m=1}^{m=N}\varepsilon^m \boldsymbol{a}_m(\boldsymbol{t})) = \Sigma_{m=1}^{m=Nn_0}\varepsilon^m g_m(,\boldsymbol{a}(\boldsymbol{t}))$ with a finite N because f is polynomial. We identify tall the coefficients of E. the recurrence begins with $(r^{-1} - f'(\boldsymbol{0}))\boldsymbol{a}_1(\boldsymbol{t}) = 0$ where $\boldsymbol{a}_\infty(\boldsymbol{t}) = \boldsymbol{a}_1(\boldsymbol{t})$ belongs to $A.S.A.P.$ Then, we have easily $(r^{-1} - f'(0)\boldsymbol{a}_2(\boldsymbol{t}) = f''(\boldsymbol{0})\boldsymbol{a}_1(\boldsymbol{t})^2 / 2!$ which verifies the recurrence. And so on. We observe that g_k are polynomials in $(\boldsymbol{a}_1(\boldsymbol{t})\boldsymbol{a}_2(\boldsymbol{t})..\boldsymbol{a}_{k-1}(\boldsymbol{t}))$ such as $\boldsymbol{A}_{n,m}(\boldsymbol{\lambda t}) = \boldsymbol{\varepsilon}^m \boldsymbol{A}_{n,m}(\boldsymbol{t})$. Degree of f being n_0, the process of identification stop for $N = n_0$.

So, the knowledge of $\boldsymbol{a}_1(\boldsymbol{t})$ corresponding to $\lambda > 1$ determines completely $\boldsymbol{a}(\boldsymbol{t})$.

C. Corollary 2

Under the previous hypothesis, if $\lambda > 1$*,* $D = 0$ *and if* $\lambda < -1$*,* $D = 1{,}25$*. If* $g = f^{(p)}$ *and if* $\lambda(g) > 1$*,* $r_\ell^{-p} = \lambda_\ell^{1+1/p}$ *and* $D = 2 - 1/p - 1/p^2$.

Now, we calculate the dimension D of each coordinate of the asymptotic solution. The dimension's formula (with the box method) is for WM ($D = 2 + \log|r_0| / \log|\lambda|$,) .

If $\lambda > 1$, then $r^{-1} = \lambda^2$ and $D = 0$, we have points only. if $\lambda < 0$, iterating f twice times, we get $g = f \circ f$. If g has the eigen values $\lambda > 1$, then $r^{-2} = \lambda\sqrt{\lambda} = \lambda^{3/2}$. So: $D = 2 + \log\lambda^{-3/4} / \log\lambda = 2 - 0{,}75$ et $D = 1{,}25$.

If $g = f^{(p)}$ and if g has the eigen values $\lambda_\ell > 1$, f has the eigen values $\lambda_\ell^{1/p}$, and the a.p. of f is $\lambda_\ell^{1/p}$. If r_ℓ is the ratio of f, r_ℓ^p is the ratio of g. The recurrence $\boldsymbol{c}(\mu\lambda_\ell^{1/p}) / \lambda_\ell^{1/p} = \lambda_\ell \boldsymbol{c}(\mu)$ of g implies $r_\ell^{-p}\lambda_\ell^{-1/p} = \lambda_\ell$.

D. Consequences

1. Mathematical

- If $\lambda = \rho e^{i\theta}$ with $\theta = 2k\pi / p$, iterating $f^{(p)}$, $\lambda^p = \rho^p > 1$, we meet the same conditions of tge corollary and the dimension is: $D = 2 - (p+1)/p^2$. We are leading to same situation as for cycles of order p. $g = f^{(p)}$ leaves invariant each point of the cycle. Invariant curves are tied to each point of cycle as previously. The solution is cyclic
- More generally, if the eigen values are complexes, $\lambda = \rho e^{i\theta}$, we can approximate θ with Dirichlet by $\theta \sim 2k\pi / p$, but we must verify the continuity of the solution with θ.
- We can replace polynomials f by analytical functions (Favard$[7]$) without changes of results: if f is diffeomorphism C^∞, interior of a disk de convergence $\|\boldsymbol{a}\| < \rho_1$, f is uniformly approximated by polynomial. Then, taking $\boldsymbol{t}$ sufficiently small, we keep the results.
-Many problems remain when we have many fixed points.

2. Methodological

We give a mathematical justification of the methods used by physicists :

- As the almost periodic function tends asymptotically to a WM's function, the use of the auto similarity is justified, with the methods of fractal dimension. We can use also auto correlation(s function.
- The sensitivity to initial conditions does not modify the global shape of the curves. The « thru» chaos results from the WM's functions.
- If we can give the global shape of the curve, we are unable to situate exactly a point on the curve at any time.

A fixed point seems to define the solution. But, if we have fixed many points in C, we have observed possibilities of passage from a domain near a fixed point to an another.

Example : case of bounded Hénon

This iteration is defined in $\mathbb{R}^2$ by

$(a_1,b_1)=\gamma a+b+h(a),\beta a)$ where $\gamma=\lambda+\lambda'$ and $\beta=-\lambda\lambda'$ with $|\lambda|>1$ and $|\lambda'|<1$. Its invariant curve is $\boldsymbol{a}(t)=(a(\lambda t),a(t))$ verifying: $a(\lambda t)=\gamma a(t)+\beta a(t/\lambda)+h(a(t))$.

If λ is negative, with 2 iterations, we are in the positive case. But, we have to study the recurrence of the twice times itered function and we have cycles of order 2 instead of fixed points and we have a fractal situation. We get :

$$(a_2,b_2)=\gamma(\gamma a+b+h(a))+\beta(\gamma a+b+h(a))+h(\gamma a+b+h(a)),\beta(\gamma a+b+h(a))$$

The eigen values greater than 1 is now λ^2. But we can have complexes roots or resonance.

Nota : In the classical Hénon's iteration, we have $h(a)=-\sigma a^2$ with $\sigma=1{,}4$, $\beta=0{,}3$ and $\gamma=-2\sigma*0{,}6313=$-1,7678. Eigen values at the fixed point 0 are $\lambda\approx-1{,}9237$ and $\lambda'\approx0{,}1559$. They are algebraic, and the theory can't be apply.

Nevertheless, if we apply it: the curve $(a,b)=(a(t\lambda),a(t))$ defined by $a(\lambda^2t)=a(t)/r=w_r(t)/r$ is: either $(1-r\beta)a(t\lambda)=\gamma a(t)+h(a(t))$, or $(1-r\beta)a(t)/r=\gamma a(t\lambda)+h(a(t\lambda))$. This last formula induces many branches that we can approximate.

4. Property P of the W-M function and resonance

Now, we speak about an interesting property of the W-M function $w_\lambda(t)=\Sigma_{k\in\mathbb{Z}}r^k(1-e^{i\lambda^kt})$. As usual, we write the ratio r as a fixed function of λ: $r=\lambda^{2-D}$.

A. Lemma

The W-M function verifies for $\forall p,q\in\mathbb{N}$ *the property P:*

$$w_\lambda(t)=(p/q)w_{\lambda^{p/q}}(\boldsymbol{t})$$

When we write $k=np+\alpha$ with $\alpha=0,1,2,..,n-1$ and $p\in\mathbb{Z}$, we have:

$$w_\lambda(t)=\Sigma_{p\in\mathbb{Z}}\Sigma_{\alpha=0}^{\alpha=n-1}r^{np+\alpha}(1-e^{i\lambda^{np+\alpha}t})=\Sigma_{\alpha=0}^{\alpha=n-1}r^\alpha\Sigma_{p\in\mathbb{Z}}r^{np}(1-e^{i\lambda^{np}(\lambda^\alpha t)}).$$

$$=\Sigma_{\alpha=0}^{\alpha=n-1}r^\alpha w_{\lambda^n}(\lambda^\alpha\boldsymbol{t})=n\Sigma_{p\in\mathbb{Z}}r^{np}(1-e^{i\lambda^{np}t})=nw_{\lambda^n}(\boldsymbol{t})$$

As we have $w_\lambda(t)=nw_{\lambda^n}(\boldsymbol{t})$ for $n\in\mathbb{N}$, then the relation is thru for all fraction p/q.

B. "Continuity" with λ

In general, $w_\lambda(t)$ is not a continuous function of λ.

But, on the set $S_\lambda=\{\lambda^{p/q}|p,q\in\mathbb{N}\}$, $w_{\lambda^{p/q}}(\boldsymbol{t})$ presents a kind of continuity in the sense:

As $w_{\lambda^{p/q}}(\boldsymbol{t})$ - $w_\lambda(\boldsymbol{t})=(q/p-1)w_\lambda(\boldsymbol{t})$, then we have: $w_{\lambda^{p/q}}(\boldsymbol{t})$ - $w_\lambda(\boldsymbol{t})\to0$ when $q/p\to1$.

C. Resonance

To study the resonance, we have to consider, for $\forall\boldsymbol{t}\in\mathbb{R}^{+d}$ instead of $\forall\boldsymbol{t}\in\mathbb{R}^{+q}$, the invariant function $\boldsymbol{a}(\boldsymbol{t})$ and find the solution of the equation E : $\boldsymbol{a}(\lambda\boldsymbol{t})=f(\boldsymbol{a}(\boldsymbol{t})),\boldsymbol{t}\in\mathbb{R}^{+d}$. Then, under H0, we search the asymptotic solution when we iterate f indefinitely. All the previous results of part II remain thru if all the eigen values are positive transcendental and no resonant. So, we study only the linear part $\boldsymbol{a}_1(\boldsymbol{t})$ of $\boldsymbol{a}(\boldsymbol{t})$ represented by W-M functions.

We begin to solve the case with two eigen values λ and λ_1 such as $\lambda_1=\lambda^{\varepsilon p/q}$ where $\varepsilon=\pm1$. In this case, we can write explicitly: $\boldsymbol{a}_1(\boldsymbol{t})=\boldsymbol{a}_{\lambda,\lambda^{\varepsilon p/q}}(t,t_1)$ in words of W-M functions. Each coordinates of $\boldsymbol{a}_{\lambda,\lambda^{\varepsilon p/q}}(t,t_1)$ corresponding to λ and λ_1 can be written as: $a_{\lambda,\lambda^{\varepsilon p/q}}(t,t_1)=cw_{\lambda,\lambda^{\varepsilon p/q}}(t,t_1)=c\Sigma_{k\in\mathbb{Z}}r^k(1-e^{i(\lambda^kt+\lambda^{\varepsilon p/q}t_1)})$

D. Proposition

If $\varepsilon=+1$*, then* $\boldsymbol{a}_{\lambda,\lambda^{p/q}}(t,t_1)=\boldsymbol{a}_{\lambda^{p/q}}(t+t_1)$, *If* $\varepsilon = -1$*, we have a resonance, then:* $\boldsymbol{a}_1(t,0) = \boldsymbol{a}_1(\lambda^{p/q-1}t,0)$

More generally, if $\lambda_1^p \lambda^n = 1$ *with* $|\lambda_1| < 1$ *and* $|\lambda| > 1$*,* $\lambda \in \mathbb{R}^q$*, we have* $\boldsymbol{a}_1(\boldsymbol{t},0) = \boldsymbol{a}_1(\lambda^{n-1}\boldsymbol{t},0)$.

If $\varepsilon = +1$, we have for the coordinates (t,t_1) corresponding to (λ,λ_1):

$w_{\lambda,\lambda^{p/q}}(t,t_1) = (q/p)w_{\lambda,\lambda}(t,t_1) = (q/p)\Sigma_{k\in\mathbb{Z}}\, r^k(1-e^{i\lambda^k(t+t_1)}) = (q/p)w_\lambda(t+t_1) = w_{\lambda^{p/q}}(t+t_1)$.

If $\varepsilon = -1$, we have a resonance.

We can take any $\lambda'_1 \in S_\lambda$ very near to λ_1 such as $\lambda'^{\gamma}_1 \lambda \neq 1$, with $\gamma = p'/q' \to p/q$ and $p',q' \in \mathbb{N} | p'/q' \neq p/q$. As (λ,λ'_1) is not resonant, by iterating f, we get asymptotically :

$\boldsymbol{a}_{\lambda,\lambda'_1}(t,0) = \lim_{t_1\to 0} \boldsymbol{a}_{\lambda,\lambda'_1}(t,t_1)$ for the a.p. λ. Then: $\boldsymbol{a}_{\lambda,\lambda_1}(t,0) = \lim_{\lambda'_1 \to \lambda_1 \in S_\lambda} \boldsymbol{a}_{\lambda,\lambda'_1}(t,0)$. We have:

$$w_{\lambda,\lambda^{-p/q}}(t,t_1) = (q/p)w_{\lambda,\lambda^{-1}}(t,t_1) = (q/p)\Sigma_{k\in\mathbb{Z}}\, r^k(1-e^{i(\lambda^k t+\lambda^{-k}t_1)}) = \Sigma_{k\in\mathbb{Z}}\, r^k(1-e^{i(\lambda^{kp/q}t+\lambda^{-k}t_1)})$$
$$= w_{\lambda^{p/q},\lambda}(t,t_1)$$

Iterating f indefinitely, the asymptotic is also $\boldsymbol{a}_1(t,0) = \lim_{t_1\to 0} \boldsymbol{a}_1(t,t'_1)$ for the a.p. $\lambda^{p/q}$.

Iterating f one time the equation E, we have $\boldsymbol{a}_1(\lambda t,0) = \boldsymbol{a}_1(\lambda^{p/q}t,0)$. Then: $\boldsymbol{a}_1(t,0) = \boldsymbol{a}_1(\lambda^{p/q-1}t,0)$.

If $\lambda_1^p \lambda^n = 1$ with $|\lambda_1| < 1$ and $|\lambda| > 1$, $\lambda \in \mathbb{R}^q$, $\boldsymbol{a}_1(\boldsymbol{t})$ verifies:

$\boldsymbol{a}(\boldsymbol{t}/\lambda, \boldsymbol{t}'/\lambda') = f'(\boldsymbol{0})\boldsymbol{a}(\boldsymbol{t},\boldsymbol{t}')$. Let $|n||q| = \Pi_{j=1}^{j=q} n_j \Pi_{j=q+1}^{j=d} q_j$. We can multiply the equation by $|n||q| : |n||q|\boldsymbol{a}_{\lambda,\lambda'}(\boldsymbol{t}/\lambda, \boldsymbol{t}'/\lambda') = |n||q| f'(\boldsymbol{0})\boldsymbol{a}_{\lambda,\lambda'}(\boldsymbol{t},\boldsymbol{t}')$. That means that we change λ in λ^n and λ' in λ'^q: $\boldsymbol{a}_{\lambda^n,\lambda'^q}(\boldsymbol{t}/\lambda^n, \boldsymbol{t}'/\lambda'^q) = f'(\boldsymbol{0})\boldsymbol{a}_{\lambda^n,\lambda'^q}(\boldsymbol{t},\boldsymbol{t}')$. Uniqueness implies : $\boldsymbol{a}_1(\boldsymbol{t},0) = \boldsymbol{a}_1(\lambda^{n-1}\boldsymbol{t},0)$.

5. Conclusion

This study is based on the fact that f is biunivoque. In other cases, the probabilistic methods previously used [5] must be applied to the iterations with multiple inverses.

But, if the number m of inverses of f is finite, we can randomly draw each inverse branch with a probability $1/m$. Then, the WM's functions can be easily randomised.

Besides, the set of the WM's functions contains some stochastic processes.

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Appendix 1. Application to partial differential equations

We consider the differential equation:

$$\partial\boldsymbol{a}/\partial\boldsymbol{x} = F(\boldsymbol{a}).$$

The unknown functions are a vector $\boldsymbol{a} \in \mathbb{R}^d$. The variable is $\boldsymbol{x} \in \mathbb{R}^{d+}$. $F(\boldsymbol{a})$ is a matrix (d,d) of polynomials of $\boldsymbol{a} \in \mathbb{R}^d$ in $\mathbb{R}^d$. With no lake of generality, we can take equal so many coordinates of $\boldsymbol{x}$ as we want. If all coordinates are equal with one x, we have an ODE. See Arnold$[1]$. First, we translate this differential equation in words of iteration.

1. Generalities

1. Notations

We call differential iteration $f(\boldsymbol{a},\delta)$ *of* $\mathbb{R}^d$ in $\mathbb{R}^d$ *the function defined as:*

$\boldsymbol{a}_1 = f(\boldsymbol{a},\delta)=\boldsymbol{a}+\delta F(\boldsymbol{a})$, *with* $\delta \in \Delta = \mathbb{R}^{d+} \cap \{\delta_0 \geq \delta \geq 0\}$.

As usual, we link δ *to* $\boldsymbol{x}$ *with* $\delta = \boldsymbol{x}/n,\ n \in \mathbb{N}, \boldsymbol{x} \in \mathbb{R}^d$. *We note* $\boldsymbol{x}F = x_\ell F_\ell \mid \ \ell = 1,..,d$.

If $F(\boldsymbol{\alpha}) = \boldsymbol{0}$, $\boldsymbol{\alpha}$ *is a fixed point of the iteration* $f(\boldsymbol{a},\delta)$; $\boldsymbol{\alpha}$ *doesn't depend on* $\delta \in \Delta$. *If* ρ *is an eigen value of* $F'(\boldsymbol{\alpha})$, *then* $\lambda = 1 + \boldsymbol{x}\rho / n$ *is eigen value of* $f(\boldsymbol{a},\delta)$.

For all small $\delta \leq \delta_0$, $f(\boldsymbol{a},\delta)$ is invertible in a neighbourhood of the fixed point $\boldsymbol{0}$; then, $f(\boldsymbol{a},\delta)$ is a C^∞ diffeomorphism. The iteration can be written in the basis of the eigen vectors of $\partial F(\alpha)/\partial \boldsymbol{a}$. In this basis, we write: $f_\ell(\boldsymbol{a},\delta)=(1+\delta_\ell\rho_\ell)a_\ell + \delta_\ell G_\ell(\boldsymbol{a}),\ \ell = 1,..,d$ where $G_\ell(\boldsymbol{a})$ is polynomial with a term of lowest degree ≥ 2.

We study the differential iteration only for $m = 1$ as in theorem 3. We rewrite H0:

2. Hypothesis 1

For all $\delta \in \Delta$, $f(\boldsymbol{a},\delta)$ *applies a bounded set* C *of* $\mathbb{R}^d$ *in itself. We have* q *transcendental eigen values* $\rho_\ell > 0$, *all the others are negative. Then:* $\lambda_\ell = 1 + \boldsymbol{x}_\ell\rho_\ell / n > 1$.

With the Steinberg series. we define d ***invariant functions*** $\boldsymbol{a}(\boldsymbol{u})$ with the equation **E**:

$$a_\ell(\lambda\boldsymbol{u}) = f_\ell(\boldsymbol{a}(\boldsymbol{u}),\delta) \quad \ell = 1,..,d$$

By iteration $f^{(p)}(\boldsymbol{a},\delta)$, we extend, beyond the disk of convergence, the definition of $f(\boldsymbol{a}(\boldsymbol{u}),\delta)$ for $\boldsymbol{u} \in \mathbb{R}^d$. $\boldsymbol{u} \in \mathbb{R}^q$ *is called **the fluctuation's parameter***.

2. Proposition 4

Under H1, the equation **E:** $\boldsymbol{a}(\lambda\boldsymbol{u}) = f(\boldsymbol{a}(\boldsymbol{u}),\delta)$ *of the differential iteration has an asymptotic almost periodic solution.*

The a.p. are $\mu = m(1+\delta\rho)^k \mid \ k \in \mathbb{Z}, m \in \mathbb{N}$. *We can write*

$$\boldsymbol{a}(\boldsymbol{u}) = \Sigma_{m\in\mathbb{N}}\boldsymbol{a}_m(\boldsymbol{u})$$

with

$$\boldsymbol{a}_m(\boldsymbol{u}) = \Sigma_{k\in\mathbb{Z}}\boldsymbol{c}(m\lambda^k)(1 - e^{im\lambda^k\boldsymbol{u}})$$

For $m = 1$, $\mu_k = \lambda^k$, *the coefficients* $\boldsymbol{c}(\mu_k)$ *verify:*

$\boldsymbol{c}_\ell((1+\delta\rho)^k) = (1+\delta_\ell\rho_\ell)^{-2k}\boldsymbol{c}_\ell$

where $\boldsymbol{c}_\ell$ *be the eigen vector of* $\partial F(\alpha)/\partial\boldsymbol{a}$ *for the eigen values* $\rho_\ell > 0$.

When $n \to \infty$: $\boldsymbol{a}_1(\boldsymbol{u}) = \boldsymbol{c}\Sigma_{k\in\mathbb{Z}}\lambda^{-2k}(1 - e^{i\lambda^k\boldsymbol{u}}) = \boldsymbol{c}\boldsymbol{w}_r(\boldsymbol{u})$ *and* $\lambda_\ell = e^{x_\ell\rho_\ell}$.

where $w_r(u_\ell)$ *is the WM's functions with* $\lambda_\ell = \exp(x_\ell\rho_\ell)$ *and* $r_{r_0,\ell} = (\lambda_\ell)^{-2} = \exp(-2x_\ell\rho_\ell)$.

Let $\hat{S}_{\delta_0}$ be the adherence of $f^{(p)}(\boldsymbol{a}(\boldsymbol{u}),\delta)$ for all $\delta \in \Delta$ when $p \to \infty$. $\hat{S}_{\delta_0}$ is closed and bounded, so $\hat{S}_{\delta_0}$ is compact. $f(\boldsymbol{a}(u),\delta)$ is uniformly continue for all $\delta \in \Delta$. When $p \to \infty$, all the coordinates of $\boldsymbol{u}$ corresponding to $\rho_\ell < 0$ $\boldsymbol{u}$ tend to 0. We denote $\boldsymbol{u}$ the vector corresponding to the *eigen values* $\rho_\ell > 0$. this variable is called fluctuation.

We get a representation of $f(\boldsymbol{a(u)},\delta)$ by almost periodical functions and we develop $\boldsymbol{a}(u)$ with Fourier's series: $\boldsymbol{a}(u) \sim \sum_{\mu\in\Lambda} \boldsymbol{c}(\mu)e^{i\mu u}$. The set of the a.p. $\Lambda(a) = \{\mu \in \mathbb{R}, c(\mu) \neq 0\}$ is countable. We apply the previous results of theorem 3:

The q eigen values $\rho_\ell > 0$ get the a.p. $\mu = m(1+\delta\rho)^k \mid k \in \mathbb{Z}, m \in \mathbb{N}$. So, we recall:

$$\boldsymbol{a}(\boldsymbol{u}) = \Sigma_{m\in\mathbb{N}} \boldsymbol{a}_m(\boldsymbol{u})$$

and:

$$\boldsymbol{a}_m(\boldsymbol{u}) = \Sigma_{k\in\mathbb{Z}} \boldsymbol{c}(m\lambda^k)(1 - e^{im\lambda^k \boldsymbol{u}})$$

We continue the computation of the coefficients $\boldsymbol{c}(\mu_k)$ for $m=1$ under the hypothesis that the $\delta\rho$ are transcendental. Let $\boldsymbol{c}$ be the eigen vector of $\partial F(\alpha)/\partial\boldsymbol{a}$ for the eigen values $\rho_\ell > 0$. The equation E becomes:

$$\boldsymbol{c}_\ell((1+\delta\rho)^{k-1}) = (1+\delta_\ell\rho_\ell)^2 \boldsymbol{c}_\ell((1+\delta\rho)^k)$$

Then:

$$\boldsymbol{c}_\ell((1+\delta\rho)^k) = (1+\delta_\ell\rho_\ell)^{-2k} \boldsymbol{c}_\ell$$

And: $\boldsymbol{a}_1(\boldsymbol{u}) = \boldsymbol{c}\Sigma_{k\in\mathbb{Z}}\lambda^{-2k}(1 - e^{i\lambda^k \boldsymbol{u}}) = \boldsymbol{c}\boldsymbol{w}_r(\boldsymbol{u})$ with $\boldsymbol{r} = \lambda^{-2}$. We observe that $\lambda_\ell = (1+\delta_\ell x_\ell / n)$. With the property II-8 of W-M function, when we take $\lambda_\ell^n | n \in \mathbb{N}$ instead of λ_ℓ, $w_r(t) = n w_{r^n}(t)$. Then: $\lambda_\ell^n = (1+\delta_\ell x_\ell / n)^n \to \exp(\delta_\ell x_\ell)$ and $r_\ell = \exp(-2x_\ell\rho_\ell)$.

3. Remarks

- If we have many fixed points in C, we observe the possibility of passages of a domain of a fixed point to an another.
- The equation of Navier Stokes $[2]$ is written as $\partial\boldsymbol{a}/\partial\boldsymbol{x} = F(\boldsymbol{a})$. Initially we have $n+1$ unknown variables (u,p) with so many variables $(\boldsymbol{x},t)$ and differential equations. We note $\partial u_i/\partial\boldsymbol{x} = \boldsymbol{b}_i$, $i=1,\dots,n$ $\Delta u_i = \Sigma_{j=1}^n \partial^2 u_i/\partial x_j^2 = \Sigma_{j=1}^n \partial\boldsymbol{b}_i/\partial x_j$, $\boldsymbol{c} = (\boldsymbol{x},t)$ $d_i = \partial p/\partial x_i$. Then :

$$\partial\boldsymbol{u}/\partial t + \Sigma_{j=1}^n u_j b_{ij} = \nu\Sigma_{j=1}^n \partial\boldsymbol{b}_i/\partial x_j - d_i + f_i(\boldsymbol{c})$$

$$\Sigma_{j=1}^n \partial\boldsymbol{u}_j/\partial x_j = 0, \quad \partial u_i/\partial x = \boldsymbol{b}_i, \ i=1,\dots,n, \ \boldsymbol{c} = (\boldsymbol{x},t)$$

If the outer forces f_i are polynomial and the differential iteration bounded, then the solution will be almost periodic as we have seen. If it is difficult to find periodic solutions $[2]$, on the other hand, the almost periodic functions solve the question. The Lorenz's attractor gives the proof. The difficulty doesn't come from differential operators which are quadratic for the iteration, but from the f_i.
- With the same methods, we can study iterations with decay: $\boldsymbol{u}_1(\boldsymbol{x}) = F(\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{T}))$: if $\boldsymbol{T} = \log(\boldsymbol{\tau})$, $\boldsymbol{x} = \log(\boldsymbol{t}/\boldsymbol{\tau})$ and $\boldsymbol{a}(\boldsymbol{t}) = \boldsymbol{u}\circ\log(\boldsymbol{t}))$, we have $\boldsymbol{a}_1(\boldsymbol{t}/\boldsymbol{\tau}) = F(\boldsymbol{a}(\boldsymbol{t})$, then $(\lambda/\boldsymbol{\tau})$ take place of λ.

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Brain Functionality via Complex Systems Theory

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Abstract. The evolution of research in the field of brain study and function has had a series of stages during the 20th century, starting with the age of great anatomical discoveries, passing through phrenology and continuing with the behaviourist and new cognitivist stages. Accordingly, in the last decades neurosciences attempted to encompass the phenomenology of psychological reality within an interdisciplinary approach. This wide interdisciplinary necessity comes from the need to apply the principles of complex systems to brain activityas well. From such perspective, it is necessary to overcome the paradigm according to which psychological activityis an exclusive product of neuronal activity. The detailed understanding of the way in which the main types of neurons function, will not help us entirely understand the mental. The theory of complex systems comes with totally different assumptions. In the complex systems generated by a great number of elements, the properties of the systems cannot be found in the sum of the properties of constitutive elements. The emergence property is the one that creates a link between the multitude of components and the properties of the complex system. As a result, even if we describe all the properties of all neurons, we will not be closer to understanding the mental.

In this paper we shall demonstrate that the psychological system has all the necessary elements in order to associate it with a complex system. That is the reason why we shall bring anatomical, neurophysiological and pathophysiological arguments, as well as data from the latest research in neurosciences using functional MRI. We shall also analyze the theories from the last century concerning the structure of the psyche in which we find elements that support a new theory of the mental from the perspective of the complex system theory.

Memorizing takes place at the interface of the spectral field with the contribution of certain information patterns as well as new information from the complex space which represents the potential, unstructured, non-differentiable, unpredictable parts. Such hypothesis is possible using a new vision on information according to which information is made up of energy patterns included in a topological dynamics.

We shall conclude that the complex space (from mathematical view point) is a real physical space and not an abstract one and that the brain dynamics between the complex space and the real one represents what we call the psyche and consists of the information processing in neural networks.
Keywords: Complex system theory; Brain; Fractals; Chaos; Topology; Complex space.

1 Introduction

The aim of this paper is to apply the theory of complex systems in order to sustain the hypothesis of the complex space as a physical space. Thus, the dynamics of the complex systems and especially that of the complex and of the real space (from the inner part of the systems) may lead to new hypotheses and theories about the structure of psyche and about its functioning.

The whole collection of the analyzers manages the transfer of information from its wave form into corpuscular form. This allows for the information processing to be accomplished both in a corpuscular, material network, the neuronal network, but also in a spectral network, of the coherent field associated to the neuronal network. Through the waves of the spectral field the dynamical link to the complex space is realized, situation which allows for the occurrence of the superior psychic processes, which are specific to the human being, and which need multidimensional development in order to be formed, a situation which is only allowed by the complex space. The mental reality represents thus the permanent dynamics between the neuronal (material) network, the associated spectral field (the fractal potential) and the infinite dimensional complex space.

The dynamics between the complex and the real space (the neuronal network) through the spectral field (wave field represented by the totality of the waves associated to corpuscles within the neuronal network) lies at the basis of the psychological system functioning. This paradigm can generate new hypotheses which should explain the mysteries of the psychological life, just as the old ”mind-body” duality. The new topical

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structure of the psychism associated with the theory of complexity and simplicity applied on fractal geometry through which reality is structured allows for the brain to have access to the knowledge of the fractal in its wholeness (when the mathematical model is reduced as a number of informational bytes, or a symbol, to put it different). Through the analysis and synthesis ability, it can conceptualize the fractal at any point and at any scale, but with the price of extensive informational data.

2 Complex systems from the perspective of modern physics

Complex systems include many components which mutually interact and which have the ability to generate a new macroscopic collective behaviour modality, whose result is the spontaneous formation of distinct temporal, spatial and functional structures. Such examples of systems can be widely frequent and can be correlated with the climate, the coherent issuance of light by lasers, chemical systems of reaction-diffusion, biological cell networks, the statistics and prediction of earthquakes, the human brain etc.

A complex system has a behaviour of an emergent type, which means that the modality in which the system manifests itself cannot be deducted from the behaviour of its components. Nevertheless, the system's behaviour is contained in the behaviour of the components, if they are studied in the context in which they find themselves. From a qualitative viewpoint, in order to understand the behaviour of a complex system, we must understand both the behaviour of its components as well as the way in which they interact in order to generate the collective action. Complex systems are difficult to study because we cannot describe the „whole" without describing each component and because every component must be described through its relation with the other components.

From a quantitative viewpoint, the "complexity" of a system represents the information quantity necessary to describe it and it depends on the details necessary to describe the respective system. In other words, if we have a system with several possible states and we want to determine its state precisely, then the number of binary numbers (bytes) which is necessary to determine the respective state is dependent on the number of possible states. The positions and the impulses of the particles are real numbers whose specification may need an infinite number of bytes. Nevertheless, the information necessary for stating the microstate of a system is not infinite. This fact is due to quantum physics, which attributes a unique value to entropy and, thus, also to the information necessary to express a state of the system. First of all, the microscopic states are undiscernable if their positions and impulses do not differ through a discrete quantity given by Planck's constant. Secondly, quantum physics indicates the fact that particles (such as nuclei or atoms) found in the fundamental state are uniquely determined by this state and cannot be differed from each other. There is no additional information which is necessary in specifying their internal structure. Under normal conditions, all nuclei are, without exception, in the state of minimum energy. The relationship between information and entropy consists in the fact that the entropy of a physical system is maximum when it is in equilibrium, thus we can infer that that the most complex system is in equilibrium state. This assertion is in contradiction with the perception of complex systems. Systems in a state of equilibrium do not have a spatial structure and do not change with the lapse of time. Complex systems have a substantial internal structure which is permanently modified as time passes.

Another challenge in the case of complex systems is the difficulty of predicting their behaviour even when the initial conditions are known, because the strength of interactions among the components of the systems completely screen the specific individual properties. It is not yet exactly known if this type of system respects some strict laws similar to the ones of the classical systems, nevertheless the development of some methods which allow for determining some of the dynamic properties of complex systems came to be possible. We should focus on representing the an-organization of complex systems which are manifested upon the passage from "complicated" to "complex" and which is based on the new paradigm of the passage from the classical space of the trajectories to more abstract spaces of the trajectories associated with the natural invariance of systems, which is characteristic to the dynamics of complex systems, which represent a separate class of entities with non-linear behaviour ([15]).

A complex system cannot be analyzed in principle by fragmentation into parts, because it is made up of elements which make sense only in within the privacy of the system. It has an unpredictable evolution, it can suffer sudden transformations which can be as big as possible, without obvious external causes; it manifests different aspects, depending on the analysis scale. It is principially different from a complicated system because the difficulty of prediction is not to be found in the inability of the observer to consider all the variables which would influence its dynamics, but in the sensitivity of the system to initial conditions (initial conditions which are slightly different can lead to extremely different types of evolution), to which the effect of an auto-organization process is to be added (process determined by the very interactions between the component subsystems and which has, as an effect, the spontaneous emergence – on principle unpredictable – of some order relations).

A complex system can be shaped and studied in an equivalent topological space, called the phase space, in which specific notions can be defined: attractors and repulsors, attraction basin, trajectories, limit cycles, etc. In this context, one can talk about functional modelling, which is a lot more abstract and ''unleashed'' of the constraints imposed by a concrete ''anatomy'' and ''physiology''. While classical modelling starts by approximating what ''is seen'', functional modelling involves the identification of an equivalent dynamical system, whose behaviour can be analyzed through specific methods, with an extremely hightened generalization degree.

In systems composed of a great number of elements, the properties of the systems cannot be found in the sum of the properties of constituent elements. The emergence property is the one which creates a connection between the multitude of components and the properties of the complex system.

3. An approach to psychism from the perspective of complex systems theory

In complex systems structure there is a potential part with chaotic aspect and a structured, causal, Newtonian part, as well as different intermediate phases. From here there results a certain uncertainty in the structure of reality. Incertitude principle of Heisenberg [16] can also be found in Gabor [14] in communication theory (the information quantum); the non-linear, potential and apparently chaotic part corresponds to the unconscious, the structured causal part corresponds to the conscious and the intermediate phases, as well as the structures which process both the information from reality and from the unconscious are represented by what Freud was calling SuperEgo. This is not only an instance of censorship of impulses and wishes with only a moral significance, but we find there the processing structures of the representation of physical reality, such as tri-dimensional vision, the synesthesia, that is the processing which structures the imaginary reality according to the capacity of our analyzers to perceive reality.

In complex systems, the chaotic part is structured through attractors according to the constraints of the system (for instance, the way some physiological needs generate, during the dream, some dream structure (thirst, hunger, sexual abstinence etc.)). These mechanisms are also highlighted in daydreaming, when the fantasies are much more adapted to the conditions of reality. Thus, there is no longer the breakage of physical laws and of causality, but only a modification of these according to subject's wish-aspiration tendency. During the wakefulness state there is a dynamics with the chaotic part, potentially unconscious in the background and which allows accessing the information, memories, the logical links (for example, a speech). Recent studies linked to the role of the unconscious when awake and monitoring the cognitive and motric activity demonstrates that there is a permanent involvement from the unconscious through different ground reactions (such as reactions of defense from a potential danger or the involvement of a psychotrauma through the unconscious in the current activity (such as blind seeing, missed facts, slips of memory, compulsive-neurotic behaviours)).

The whole cosmologic and biological evolution is resumed to a dynamical link between chance and necessity, between diversity (chance mutation) and selection, between chaos and structuring, as in the human body (permanent renewal of cells and tissues, as well as the dynamics between inflammation (disorder) and structuring). Thus, old age, disease, epilepsy, rhythm troubles can be interpreted as losses of the fractal character, through the reduction of the chaotic character.

Information represents codified energy which is expressed under the form of patterns, structure patterns, initiated by attractors which activate in the phase space, between the chaotic and the structured part. The information is stored in the spectral space and expresses the patterns in the structure of atoms, molecules, macromolecules and cells. It has a potential existence which is expressed through substance and energy in certain conditions (of local coherence).

The virtual projection from optics or from projective geometry can be associated, so that when the whole physical (Newtonian) reality to which we have access through our sense organs, through perception, represents a projection in the imaginary space. We could thus build a mathematical model of this space using imaginary numbers, complex (imaginary) geometry, imaginary time, topology etc.

A virtual, Newtonian reality as projection of physical reality is completed by the unstructured, a-causal, apparently chaotic component: the imagination, the dream, the failed acts, the subliminal mechanisms, the unconscious etc., which can be associated with the a-causal, potential, unstructured and non-differentiable component of complex systems, the source of inspiration, of creation and of access to non-euclidean realities to holospace. These potentialities can become conscious through patterns (see the archetypes and the collective unconscious of Jung [17]) and they can be found in logical, algorithmic, organized and systematic form in everything that is creation (from making a speech, conversation, improvisation, to creating new musical pieces, new artistic work, new scientific work). The chaotic, unpredictable part does not only contain the Newtonian reality to which we all have access, but much more, maybe even the structure of the whole Universe, at potential informational level. The brain has access to the implicit part (the implicit reality of Bohm [9]), if we associate this part to what the classics called unconscious. From here derives the capacity for mathematical reasoning, for physics, for reasoning reality in n-dimensional spaces, a-temporal realities, a-spatial realities.

The fractal geometry of reality confirms the older intuitions connected to the structuring of the Universe, which would have the same functioning and forming principles, irrespective of the scale. The physics of black holes and the astrophysics of the last years, as well as the theory of Big Bang, have brought arguments to support the idea that the fundamental principles of quantum mechanics can be found in the structure of the Universe.

By continuing to look for elements in order to sustain the unity of the Universe, it is necessary to analyze the theory of complex systems and, also connected to it, (as a physical approach), the complex functions or the complex space from a mathematical perspective. The complex analysis is absolutely necessary in describing the spinning movement, including that of the magnetic vector from the electromagnetic wave, as well as in the fluid dynamics.

Complex space could then describe a physical reality which integrates newtonian reality, as well as quantum mechanics or cosmology. For instance, Yang [21] considers the complex space as a physical entity, in which one can describe an entire variety of phenomena, among which one can find classical mechanics or relativistic mechanics.

The unpredictable, a-causal, unstructured part, which is potential in the complex systems structure, can be found in the structure of the spectral field associated to the corpuscle from the structured, causal, newtonian, predictable part. This spectral component contains, through the imaginary component of the wave formula which describes the phase (the dynamics of the magnetic vector), the access towards the complex spaces, where the whole information is to be found, as it is structured in the topological geometry of the energy configurations. The infinite dimensional possibility of these complex spaces, just as the infinite diversity of topological transformation within these spaces, together with their scale invariance allows for the estimation that in this complex space which is dimensionally infinite we can have access to all the information in the Universe.

Thus several theories are gathered together in a unitary approach: the theory of complex systems, which comes from a physical perspective in the physics of the fluids, the fractal theory, the theory of chaos and topology, with the complex analysis and the complex functions which use complex numbers with their imaginary component and which describe, in physics, the imaginary, unpredictable, potential, non-differentiable part, which can be found in the theory of complex systems. The semantic confusion, the apparent different significance of the word *complex* within the two theories or approaches is proved to be, on the contrary, a coincidence which is not random, but is connected to the synchronicities of Jung.

As in Mathematics the information can be stored or processed by algebrical equations or by trigonometrical functions, also in the physical reality information can be structured either algebrically or geometrically. The Fourier series and the Fourier transform achieve this through the interface between a spatial and temporal reality and a spectral reality. Because spectral reality is a-temporal, a-spatial, the Fourier transform and the reverse of the Fourier transform make this switch between the algebrical description and the geometrical one. The mathematical model for complex spaces includes the existence of topological transformations in an infinite dimensional space. As a result, the reality of the wave formula as being a-temporal, a-spatial, it represents an interface between the Newtonian reality and the complex ''reality", that of complex spaces (Hilbert space).

The dynamics between the complex space and the physical one is an expression of the mathematical description of reality by algebrical or trigonometrical equations. The potentiality can be encompassed, codified in trigonometrical equations and it expresses the information in an a-spatial, a-temporal reality, which is specific to the wave and is algebrically transformed into a geometrical form when a spatio-temporal reality emerges, as it happens when the wave is collapsed into a corpuscle. In both cases, topological transformations are possible (and in an a-spatial, a-temporal situation which is trigonometrically expressed, but also in a spatial and temporal situation which is algebrically and geometrically expressed).

The discontinuity of reality which is described by Planck as an energy quanta, by Gabor in information quanta, the non-differentiability which is specific to fractal dynamics, just as the property of complex systems together with deterministic chaos, all are due to a continuous interference between the physical reality and the complex one, by means of spectral field. Depending on the local field conditions, of force field and of scale structure, under the action of atractors, the information (patterns of qualitative energy, diversified through topological transformations) is taken over in order to structure the quantum or cosmic Euclidean space.

4. The imaginary space as a physical and mathematical reality, from a psychological perspective

Complex systems can be identified at different scales, a method which can be applied also to the imaginary space. In the imaginary space, time has a spatial dimension property, which allows for movement in both senses of its axis. The construction of this imaginary space is made by the same methods as the ones used for the space of physical reality, although it has additionally elements which elude to it, that is the implicit reality of Bohm, such as, for example, n-dimensional spaces, fractal developments beyond what we can find in physical reality, plus mechanisms specific to deterministic chaos and generally speaking to complex systems.

The complex analysis is essential for the description of physical reality, of spectral, wave, field phenomena, which together with the corpuscular ones contribute to building the physical reality. The electric field corresponds to the real part, whereas the magnetic field corresponds to the imaginary component. The magnetic vector has a rotation movement around its own axis, movement which is described by the complex systems. At a 90-degree rotation (multiplication by i), an inversion of the components of the complex number occurs, event which in physics involves a Wick rotation. By multiplication with i, frequency and phase are mutually modulated, and their correlation is achieved by means of information.

Complex analysis describes physical phenomena which take into account the spinning movement. This phenomenon is present first of all in the electromagnetic waves and thus it can be found in many theoretically and technologically described situations. If we accept that there exists, in the real world and also in the functioning of the brain a spectral, wave component, then the description of the phenomena at this level requires the use, in mathematical modelling, of complex numbers with their imaginary part, of complex plans and so of complex spaces. Thus, the imaginary space, which encompasses the space of psychic activity, can be described by complex analysis, so that the syntagm ''imaginary space" is not only a metaphor, but a real physical space.

All these are associated in the description of different physical realities and phenomena, which are, in one way or another, connected to the spectral reality of the field and wave associated to every particle. Surprising as that may sound, these complex spaces coexist with our Newtonian reality, as they are present in our every-day reality, as we are delved into a spectral, electromagnetic reality, to which we are closely-linked. As a consequence, a reality co-existing with us is the a-spatial and a-temporal reality, described by the wave formula and which is involved in the phenomenon of visual perception, in which the undertaking of spatial and temporal information is achieved by light through the modulation of its frequency, a phenomenon which is described by the Fourier transform, while the stimulation of the retina involves the collapse of the wave formula and the emergence of corpuscles which stimulate cells in the retina through the reverse of the Fourier transform. As a result, all we look at and all we see, in order for it to be seen, passes through an a-temporal and a-spatial phase, within the time lapse which is necessary for light to reach us, coming from that object. This lapse can be millions of light years for cosmic objects, or infinitely small second fractions when we look at our friends, our house or our garden.

The imaginary time represents only one of the dimensions of the imaginary space, the other ones being spatial dimensions which can be described as imaginary dimensions of the complex space. At small distances, at speeds within our Newtonian space, time can be seen and represented as a size which measures the succession of some events or the interval between them. If we use the equations of the relativity theory (space-time continuum) for very long distances (the distance Sun-Earth and the light velocity), then the time resulted from these formulas is described by a complex number, with a significance of imaginary time. This would lead to the conclusion that practically speaking, we as people use only this imaginary space, or, to put it differently, our representations of time actually use the imaginary time in Einstein's relativity theory. This imaginary time, or the time from the imaginary space is a time which, as compared to the Newtonian reality, does not have a single sense. In the imaginary space, time has the characteristics of a spatial dimension, as it can be run in both senses, in the past and in the present.

If in the space of physical reality, time is run in only one sense, because of the dynamics towards an increase in entropy triggered by the Big Bang, in the imaginary space it seems that it makes an enclave, a break from the cosmic dynamics of the Universe expansion, as long as evidently, in our brain we can evolve in living and updating the past, but also construct variants of the future. Without this possibility, neither memory nor the conscious action oriented towards the aim would exist, there would not be psychological life as we know it, as the neurological studies have demonstrated that without memory neither new experiences could be assimilated which are based on old ones, nor coherent and focused actions could be achieved, if they need the experience of the past.

For almost a century it is known about the existence, on the cortex, of projections of the sensory and motric structure of the body of that which is classically named sensitive and motric humunculus. Research on psychopathological situations such as the situation of the syndrome of the phantom limb, bring arguments on a spatial projection at brain level of every segment in the body. The fact that this cerebral representation of the segment remains functional for a longer or shorter period of time demonstrates both the existence and the persistence of such representations.

The mirror box technique applied by Ramachandran [18] for the persistent, painful and spasmodic phantom limb cases shows that the representations of the segments of the body have a spatial character, as long as they can be influenced by the illusion of topological modifications, outside the imaginary space. The fact that the cerebral image of the lost limb segment persists away from the normal period after an amputation shows that some circular reverberant circuits maintained by remembrances marked by pain, contraction and suffering, are involved in the persistence of this structure which is spatially cerebral. These experimental facts lead to the conclusion that in the imaginary space there is a projection of the spatial structure of our body, to which it participates along with sensoriality and motricity, with the sensory organites and the corresponding neuromotric plaque and the affective, positive or negative processes. In fact, Davidson [13] demonstrated in his research that affection is involved in all the cognitive processes, including in the projection of the body and of the whole reality, at the level of imaginary space.

On the other hand, at an overview, the phenomena of suggestion and suggestibility
from the modern theories point of view are involved in the Ramachandran [18] technique of improving the residual or complicated phantom limb symptom. A whole series of studies have demonstrated that we are willing to accept and to believe, as long as there is a motivation, be it affective-emotional or even logical, rational. In order to be able to reconstruct the action of a book or film, of a speech or of a lecture, it is necessary that, in our brain, we have a virtual reality, an imaginary one, which describes what in fact we call imaginary space. In the last decade the so-called mirror neurons have been highlighted and they recently acquired scientific validity through research with functional RMN and which brought objective proof for the existence of a virtual or imaginary projection of the Newtonian geometric space in which we live. Excitation of these neurons in the motric, sensitive or sensorial area to the actions and the behaviour of the others comes to sustain the previous so-called theory of the mind, which was trying to explain our ability of intuition, of perceiving the feelings and thoughts of the other. Mirror neurons come as objective arguments which sustain this theory, which was explained previously by psychologists as being a result of relationships with the others, communication and our specificity as social beings. They also represent a proof of the existence of spatial and temporal structures in our imaginary.

5. An explanation of psychism from the new paradigms perspective

Complex space, which is considered to be a purely mathematical, imaginary, abstract one, can actually be a physical space (without which quantum physics would not have any coherence any more) and which includes the real space which it generates and maintains through permanent dynamics. This change of paradigm is also important for macro reality from our Newtonian level and even cosmical, through the theory of scale relativity and, just as we described before, by interfering in the dynamics of complex systems through the scale invariance of fractality and topology. Thus, the notion of ''complex" in the complex systems theory conceived in order to describe the systems with an indefinitely high number of elements in order to distinguish them from the complicated ones gains a significance which overlaps the one in the mathematics of the complex space.

Tegmark [19] maintains that mathematical structures and the relationships between them lie at the basis of reality. The elementary particles themselves are mathematical structures which can be perfectly described only by mathematical properties; all these form something that we generically call information. Another argument of the physical character of the complex space is the description of the wave function and of the wave function equation, which impose the existence of the Hilbert space. This abstract space allows for the inclusion of both the real part of the wave but also the imaginary, complex part of the wave (Schrödinger). This space requires the inclusion of both the real part of the wave, but also of the imaginary, complex part. As a result, the Hilbert space has properties of the complex space (the infinite dimensional character), the description using complex analysis, but also the real part which includes the wave amplitude and the potential capacity of becoming real in the collapse of the wave formula.

Another element which belongs to the real part is the space-time continum, which we find in the Minkowski space, but which we also find in the Hilbert space concentrated in the expression of characterizing the wave as being ''a-spatial", ''a-temporal". In our view, the Hilbert space is an interface between the real and the complex space and a proof that the complex space is a physical space connected through a permanent dynamics with the real space, as long as we accept a wave as being real, with its wave function and equation.

The very notion of complexity needs also another approach. From the general theory of systems from the 60s conceived by Bertalanffy, in the last decades, the theory of complex systems or the complexity theory are more and more mentioned, as they include a whole series of theories which imposed in the last decades (the chaos theory, the fractal theory or fractal geometry with non-linear dynamics, non-differentiability and topology).

All these theories are trying to describe as close to reality as possible the intimacy of the systems functioning with a huge number of elements, which interacts with other systems (dissipative systems) and which in fact can be found anywhere in the physical reality. These systems have a series of properties, among which emergence is a property with special implications, but also the dynamic structure they presuppose, generally characterized by a structural, causal, Newtonian, predictable component and another impredictable, a-causal, non-structured, potential component. Physical experiments (the ones in the plasma tubes but also in the dynamics of fluids, etc.) highlighted these components as well as the dynamics between them, which presuppose a tendency of auto-structuring through the attractors, within a space called the phase space. There is still an important question related to the source of information which allows for the auto-structuring and thus the dynamics between the potential and the predictable component. The current explanation for the source of this information is that it comes from the privacy of the system. However, in the structure of the system (if we remain at the more simple model of the plasma tube), there are only particles and their attached wave component. If we consider that the information contained by the particle comes from the coherent wave, the obvious question is where the information at wave level comes from. Currently, in every day life, in the information technology era, the information is transmitted via waves, by means of their analogic transformation into waves which modulate a carrying wave. Modulation can be the amplitude modulation (little used because it is too easily affected by noise, but anyway the amplitude is in inverse ratio to frequency), the generally-used way is that of angle modulation, which means modulation of either the frequency, or of the phase, which is transmitted in the end to the modulation of the magnetic vector angle. The phase is recognized as being an imaginary, complex component of the wave formula. The movement of the magnetic vector, described by the complex equations, generate a complex plan, which connects the wave to the complex space, which allows for the storage of information in the topological modifications from this infinite dimensional space. To put it different, the information in the complex systems is to be found in the complex space, which renders the potentiality, non-differentiability, a-causality characteristics from the description of complex systems ([1-7], [11], [12], [14]). Coming back to the plasma tubes, the intimacy of the system from where the information comes is represented by the coherent waves phase with every particle (wave corresponding to every particle from the wave-particle duality), which represents the connection to the complex space, where it can be found at the potential mode, as information, the whole physical reality. According to the constraints of the system from the complex space, through the wave phase, the information which reaches the particle that generates the auto-structuring pattern is undertaken.

The topic of the dynamics between the two components (the structured, causal, differentiable, Newtonian component and the potential, unstructured, a-causal component) is to be found in the psycho-analytic conception over the psychological system (see also [10]), which is then repeated under different forms in the theories of psychism, namely the unconscious (id), subconscious (superego) and the conscious (ego). The unconscious represents the unstructured, a-causal, potential, unpredictable part which we can highlight in what we can call dreams, failed acts, lapses (as Freud himself describes), and the structured, causal, differentiable and Newtonian part is what was called conscious. In the psycho-analytic view, the super-ego is considered to be partially conscious, partially unconscious and it contains (according to Freud) the totality of the norms, rules, social laws, moral laws, which are constructed in the psychological space through education, as they represent elements with a value of law, faith, the nucleus of convictions through which the environment information is processed. From the complexity theory viewpoint, this superego could be associated to the phase space, where these convictions and values help with processing the information in conscious mental structures. Compared to Freudian theory, the theory of complexity would suppose that, at this level (superego) there are not only the moral and social values and norms, but also the processing patterns of the Newtonian laws connected to space, time, movement, just as the other rational precepts which science offered to the modern man in order to help one adapt to the environment.

The analyzers achieve, on principle, the transformation of wave information in the corpuscle, thus generating the tri-dimensional and spatial-temporal vision of reality, but the processing, at brain level, is also spectrally-made (de Valois [20]).

From a physical viewpoint, at any scale, there is a differentiable hidro-dynamic description mathematically modelled through hydro-dynamic equations, but also a stochastic, potential description, expressed through the equation of the wave formula. If we accept that the Hilbert space presents both the properties of the Minkowski space as well as those of the Euclidean one, but also of the infinite dimensional complex space, then it results that the Hilbert space represents the interface between the real space with all its descriptions and the complex space with its whole potentiality. Thus, the whole psychological life can be considered to be developed in this Hilbertian space which allows also for a Minkowskian perspective, a spatio-temporal continuum, under the form of fractal space-time, where the trigonometrically-stored information is a-spatial and a-temporal, thus creating the conditions of a stable memory, but also a tri-dimensional spatial and temporal perspective which represents sections in time and space of this continuum. This material component of the neuronal network allows for the processing of information but for superior psychological processes, the processing is made in the complex space, so that the synthesis, generalization, abstractization and conceptualization suppose a multi-dimensional perspective, which can be achieved only in the complex, infinite dimensional space.

What we find, at quantum level, described by the Hilbert space, the real component, along with the fractal space-time and the complex component, at the level of the brain the interface of the neuronal network, the spectral field and the complex space, at a cosmic level, the Euclidean and Minkowskian spaces, together with the Riemannian one, connected to the complex space which from now on will be called matter and black energy. This hypothesis follows the principles of fractal development, which remains scale invariant.

On the other hand, as specialists in neurosciences sustain, just as the anthropologists, a radical qualitative leap for the development of the human species was the emergence of mirror neurons. They are present in other mammals, too, but it seems that in the case of human beings, through a genetic modification, they reached a degree of numerical development or maybe qualitative development which made this leap possible; it was expressed through a radical development of the social life, but which most of all permitted the transmission of information, abilities and behaviours, within the same generation and which, being transmitted to future generations, gradually constituted what we call today culture. The mirror-neurons which were highlighted about 20 years ago were recognized as being present at humans in the last 10 years, with the help of functional MRN. The study of these neurons is still ongoing, but just as the wave-corpuscle duality of one century ago, mirror neurons also start to raise some epistemologic problems. They allow for a connection between the subjects in a relationship, a connection which explains, for example, ''the old theory of the mind", built by the psychologists a long time ago in order to explain the empathy, compassion and intuition phenomena of the feelings of others. There remains a great problem, connected to the physical way in which mirror neurons are connected, in one person or another, especially because the last researches highlight the fact that the involvement of the visual sense and of other senses is not necessary, as long as the stimulation of mirror neurons with an individual is achieved by the intentionality of the action of the other individual. It may seem that a form of communication is involved, discussed until now more in the sphere of parapsychology, but which could find a scientific explanation in the dynamics of the psychological system between the neuronal network, the spectral field (the fractal potential) and the complex space.

The processing of information is made for the information supplied by the analyzers in a differentiable, causal, algorithmical form at the level of the neuronal network, whereas outside the analyzers, within a complementary network found in the complex space, mediated by the fractal potential from the spectral field of neurons. As a result, the qualitative leap represented by the emergence of culture would not be generated only by the emergence of mirror neurons which are present also with other animals, but by the development of genetic patterns which allowed for a better connection between the two networks. Not randomly, the appearance of articulate speech is associated with this qualitative leap in the development of humans. The speech centre seems to represent a system of information processing which allows for the connection to the infinite dimensional and complex space and thus the possibility of superior psychological processes. The study of mimic and gesture language of individuals with deafness highlighted the fact that, when learning this language, there is a first phase of learning of a mimic and gesture behaviour which is processed in the right hemisphere, specialized on spatial representations, and it becomes a real language only when it is undertaken by the speech center from the left emisphere. Then the mimic and gesture is undertaken at a level of notions and concepts and superior processes of abstractization, synthesis and generalization can be achieved. It results then that the centre of speech can be such a module which allows for the connection of the neuronal network with the corresponding one from the complex system, thus explaining the leap towards Homo sapiens. The centre which demonstrates the connection with the infinite dimensional spaces of the complex systems is the centre of speech (the deaf and dumb language), music processing and intuition, imagination, the ability to know some realities beyond the Euclidean space.

In the field of knowledge, fractal theory highlighted the fact that, in spite of the apparent infinite complexity of reality, this is in fact built on the basis of a fractal geometry in which the iteration of an extremely simple structure or configuration (the generating equation) combined with the topological modifications at every dimensional leap can reduce this whole complexity for a fractal collection which could theoretically lead to a single fractal, to a single configuration, to *One*.

Tegmark [19] proposes that this immeasurable complexity is generated by our approach of an extremely reduced sector from the scale section of a fractal. This is the aspect of reality that the positivist is trying to get to know through the scientific experiment. The analysis and description of this aspect of reality needs an immense number of informational bytes and at this level the generalization and abstractization capacity of mathematics allows us to build models to approximate this reality. We started from the premise that science researches an extremely reduced sector of the scale section of a fractal. If we consider the whole fractal, all is reduced to a simple equation (Mandelbrot's equation $F(z)=z^2+c$). It seems that we have the possibility to represent our reality also at the level of complete fractal. It is what mystics, philosophy and metaphysics did for milennia. The place where the undeployed fractal can be found, where the ''wrapped" reality of Bohm is, is the complex space, where there is the whole reality envelopped in potential under the form of mathematical structures which represent the equations of fractal generation. The first form of deployment of information from the complex system takes the form of the energy we find in the physical field under non-differentiable continuous form, but also at quantum level and at Minkowskian level. The next deployed form of the fractal is to be found under spatial and temporal form, under corpuscular form at quantum level or Euclidean form at tri-dimensional level. The representation of knowledge - through its scientific theories but also the philosophical and religious concepts - consists of two complementary aspects which physicists of a century ago presented under the form of the wave-corpuscle duality, while those of current day give a differentiable description which is mathematically modelled through the equations of hydrodynamics, as well as a stochastic, potential description expressed through the equation of the wave formula.

6. Is brain a computer?

The neuronal network development is made on fractal criteria, just as all the other apparatuses and systems of the human body. In the brain, the transmission of sense perception is spectrally and vibration-achieved [20]. As a consequence, the spectral field formed by the waves corresponding to corpuscles from the neuronal network are coherent, allowing for the processing of information both in the neuronal network and in the spectral space (Hilbert space), where at any scale there are the two types of realities, a differentiable and a non-differentiable one, highlighted through the hydrodynamic model of Madelung and the stochastic model, respectively. The a-spatial, a-temporal component allows for memorization, whereas the complex component offers the possibility of multi-dimensional processing which can explain superior psychological processes, such as, for example, conceptualization, semantics, abstractization and generalization, etc.

As opposed to the electronic computer whose hard structure is structured after some artificial algorithms (Barabassy [8]), the spectral component corresponding to corpuscles from the hardware has the same artificial character, deprived of the fractality specific to natural development, as a result there is no coherence between the substance corpuscle network and the spectral wave one.

Another essential difference between the electronic computer and the human brain is given by the analogical specific of the psychological processing, as opposed to digital processing. Analogical processing is doubled by the configurative topological character of the processing, practically speaking it is not numerical processing or only numerical processing, it is also a processing which belongs more to topological geometry. The dimensional dynamics from the 0 dimension to the infinite dimensional, which in our reality is achieved only up to three dimensions, can be achieved in the psychological reality in a multidimensional way in the complex space (through the fractal potential).

In the structure of psychism, the access from neuronal network to spectral (fractal) field and through Hilbert space to complex space allows for multidimensional dynamics which is not met at the electronic computer and which can explain superior psychological processes such as conceptualization, semantics, abstractization and generalization, etc., but also what is specifically human, creativity, intuition and adaptability.

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An Approach on Information from Topological View

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Abstract. To define information is not easy task due to the diverse forms in which it can be expressed and identified. The main forms that occur (data, information and knowledge) do not represent a mere structure with increasing complexity which implies the integration of information in knowledge and that of data within information. For data to represent information a processing system is necessary. For information to construct knowledge, the human psychic is necessary. On the other hand, Shannon's theory which is the basis of informational phenomena implies the approach of information from quantitative view and less from a qualitative one.

We shall demonstrate that this qualitative aspect is generated by the topology of the geometrical space which, in its turn, organizes the informational dynamics and explains the unity of reality from the informational point of view due to scale invariant feature of topology. We shall argue that from the qualitative point of view, information is made up of energy patterns situated at different topological configurations, while according to the quantitative approach, besides entropic elements, information is implied in fractal dynamics, the topology of geometrical space interfering in dimensional change. Such hypothesis will be supported by implying topology in all scales and reality levels, using the string theory and quantum physics, a new perspective of wave-corpuscle duality, as well as considering the molecular, biochemical, biological and mental levels, i.e. those places where information is permanently retrieved within topological dynamics.

We conclude regarding the hypothesis according to which topology as a mathematical discipline applied on information at different scales can offer a coherent perspective and an answer to the question "What is reality?"

Keywords: Information; Topology; Complex system theory; Fractals; Chaos.

1. Introduction

In our paper, we want to treat the information correlated to the substance and the energy, by applying the theory of complex systems, of complex analysis and of topology. We aim to highlight the fact that information can be found in the complex space of the wave phase spectral field. As a result, this complex space can be found anywhere and at every level of the reality. In our view, it is infinitely dimensional, as it can contain all the information in the Universe. From a mathematical viewpoint, the real space is included in and intertwined with the complex space generated by the electromagnetic waves. At quantum level, this intertwining can be achieved by the collapsing of the wave formula into the complex space of the wave phase and it can be transmitted into the complex space of the spin rotation, by transferring the whole information. This phenomenon is specific to reality at the level of the whole knowable universe, as everywhere there are electromagnetic waves and also at every level of the reality, including the human brain.

Our hypothesis is that the complex space is a physical space, which includes the real space which it generates and maintains through permanent dynamics. Thus, the complex space describes in fact a physical reality which integrates Newtonian reality, quantum mechanics and cosmology etc.

2. Information. Definitions and concept-making

In an etimological sense, the information is what gives shape to the spirit. It comes from the Latin verb *informare*, which means "to give shape" or "to form an idea on something". The perception on the information is as heterogenous as possible, the concept of information being a subject for reflection and analysis in: information theory, communication theory, knowledge theory, logics, semantics, philosophy, theology etc. Mainly, data forms information and information constitutes knowledge. Actually, the phenomena is not reduced only to an inclusion of a field into another. The information needs data and operation and memory systems, whereas knowledge supposes an accumulation of information, but also of superior psychological systems, such as generalization, abstractization, synthesis, correlation and significance. This diversity under which information is presented determines both the defining difficulty and a unitary understanding of its significance at different levels of reality.

With quantum mechanics, the necessity emerged to define information at quantum level. In the theories which appeared in the second half of the twentieth century (the theory of chaos, the theory of fractals and of non-linear dynamics), all united into what is called the theory of complex systems, the necessity to define information appears more imperatively, especially because this theory is applied irrespective of the scale, to all levels of reality. The science of complexity, which attempts at modelling the structure of matter at different scales or reality levels, needs a new approach of information as a defining notion along with energy and substance. This is the reason why defining information becomes even more complicated from the perspective of the new paradigms. Traditionally speaking, there are two meanings of the information notion. One with the aristotelic acception, which designates the formation and structuring of a specific form, of an organization within an initially non-homogenous matter, the other signifying the transmission of a message. Information can also be seen as a proper fact, as a relation fact, as a fact of action transmission. That is why we are talking about an objective information transmission which is related to the structure of the Universe, be it macroscopic or microscopic, but also of a subjective meaning, which involves human communication, not only between human beings, but also between them and the various information technology devices and technologies.

The theory of information is connected to Shannon and Weaver [21], who defined, in the 50s and 60s, information as an entity which is neither true nor false, neither significant nor insignificant, neither credible nor doubtful, neither accepted nor rejected. As a result, it is not worth studying anything else than a quantitative component of information, but not also the semantic part, which allows for the association of information with the second theory of thermodynamics, with entropy, the information or the quantity of information being in inverse ratio with it.

Weaver connected Shannon's mathematical theory with the second thermodynamic law and asserted that entropy is the one which determines the information generation ratio. The formula of information is identical to the one of entropy elaborated by Boltzmann, but considered with a minus sign:

$$\mathbf{H = -\sum_{k=1}^{i} p_k \log p_k}$$

where p represents the probability of an element or event k within the system.

Information is, thus, entropy. It is important to notice that Onicescu [17] also formulated the hypothesis regarding the fact that the degree of organizing a system can be " measured" with the help of informational energy, thus defined:

$$\mathbf{E = -\sum_{j=1}^{n} p_j^2(A)}$$

where p represents the probability of appearance of the event A.

This quantitative approach of information is applied in the field of telecommunication and of information technology. Under this approach it is important to establish the quantity of information and its true or false character in transmitting information, to which probability notions can be connected in order to find, with the receptors, the source-transmitted information. Even within this technological approach, two aspects of information are highlighted: information as a product, which reflects a static overview, and the approach as a process, which highlights the genesis and the scope of information. In fact, the two aspects represent the information as potentiality and the information expressed and involved in the dynamics of the becoming and structuring of matter.

Upon attempting to structure the multiple informational approaches, Introna [15] distinguishes two archetypes: the informational and the communicational one. The first was patented with the explosive development of informational technology and is connected to the making (development) of "productive" informational systems. The second has its origins in the communicational frame of Shannon and Weaver [21], being less important in the field of informational system field, but it is more widely accepted in the theories of communication. Similarly, Stonier [18] is of opinion that the fundamental aspect of information is connected to the fact that this is not a mental construction, but a fundamental property of the Universe. Any general theory of information must start with the study of the physical properties of information, as it is manifested in the Universe. This action must be taken before attempting to understand the variants and the more complex forms of human information. The next step must involve the examination of the evolution of informational systems beyond the physical systems, first in the area of biology, then in the human, cultural area.

The scientific approach of the information theory starts from the classical opinion that mathematics is the general language of nature. The structure of the Universe is written in the mathematical language, and its letters are geometrical forms, symbols and mathematical relations. Tegmark [19] maintains that at the basis of reality there are mathematical structures and the relationships between them and that elementary particles are mathematical structures which can be perfectly described only by mathematical properties. Thus, these mathematical structures and the relationships between them define what we call today information, whereas science does not do anything else but decypher the information contained in the structure of the matter, by physical-mathematical modelling. According to this paradigm, information is to be found in nature, outside of, beyond and independently of the observer. As a consequence, information must have existed before the appearance of human conscience.

To put it different, the information is the fundamental component of reality, such as matter and energy, as the nature is filled with information. On a larger scale, information exists before, or, in other words, knowledge is ”more fundamental” than its observer and interpreter. Thus, the reunited concepts of matter (substance and energy) and information can explain the emergence, the forming, structure and dynamics of mind and knowledge, but also of the whole structure of the Universe. Information has an objective natural existence; people absorb it in their minds and the computer memory modifies and multiplies it through thought and bring it to the ”middle” of society via the language.

At the opposite end of this materialistic-objective approach of information is the belief according to which information is something one person communicates to another, whereas the meaning of information can be understood only if we take into account the presence of alive beings endowed with reason, placed into a socio-cultural context and analyzed from a historical perspective.

A fundamental trait of information is connected to its subjectivity. Whatever can be information for a person can mean nothing to other people. Whatever is considered as information for a person can be data for another person. On the other hand, starting from the same set of data, different individuals, through different processing, can infer different information. If the data has a physical, tangible existence, the information exists only with the receptor, thus it is intangible. Information is the product of human or artificial intelligence and what constitutes information for one person can represent mere data for another person.No matter how difficult the definition and significance of information is, a possible modality of understanding what information represents in its essence is to be able to define the connection between energy, substance and information.

3. The place of information in the wave-corpuscle duality

The paradoxes highlighted by quantum mechanics in the first half of the 20^{th} century include, apart from the uncertainty relations of Heisenberg [14], a strange involvement of the observer in developing quantum phenomena. These facts suggest that the splitting into subjective and objective information is artificial and that they should be regarded as aspects of the same phenomenon. In order to uphold this idea, we must take into consideration another paradox of quantum mechanics, which is just as exciting and linked to the entaglement phenomenon, which, as a result of repeated experiments, highlighted a reality which is hard to infer, that is that all the particles which interacted at a certain point remain connected.

All these paradoxes that quantum mechanics imposed, along with the wave-corpuscle duality, determined a new approach in physics, mathematics and in the scientific approach in general. If during the 20^{th} century it was studied from the elementary particles’ point of view, of the wave component from the spectral viewpoint and materially under the form of substance and energy, the information was not treated at its true value, according to the role it has in quantum mechanics. The information technology era, as well as the theory of complex systems, with the chaotic aspects in which information has a potential character, but which explains the dynamic evolution patterns of the system which is highlighted in the phase space, have all imposed the comeback on the role of information at quantum level.

The complex systems theory imposes re-analyzing the wave-corpuscle duality from the perspective of fractal geometry and of non-linear dynamics, which also need the involvement of information as a third element in the wave-corpuscle duality.

In Scale Relativity Theory, the dynamics of any physical system is described through dimensions which can be expressed through fractal functions, that is functions which are dependent both on coordinates and on time, but also on resolution scales. Moreover, any quantity can be written as sum between a differentiable part, i.e., dependent only on coordinates and time, but also on a fractal part, i.e. dependent on both coordinates and time, but also on resolution scales. In such a context, the differentiable part is proved to be compatible only with the predictable states of the physical system, while the fractal part is proved to be compatible only with the unpredictable states of the same physical system.

The analysis of wave-corpuscle duality in de Broglie's theory involves the simultaneous existence of two types of movements: a deterministic movement, which is predictable and associated to a continuous movement of hydrodynamic type along a continuous line, which is specific to the corpuscle character, and a zig-zag random and unpredictable movement, which is specific to the wave character. De Broglie's model introduces the two types of movements only as hypotheses, but the real problem, how much it is wave, how much corpuscle, as well as the wave-corpuscle structural compatibility (the structure of the wave should be compatible with the corpuscle structure) has not been solved yet.

A new approach modality of the problematics involved in the wave-corpuscle duality resides, in our view, in supposing that the movement of a particle takes place along continuous and non-differentiable curves. This means passing from a classical approach of movement within an euclidean space to a non-conventional, non-standard approach, with the assumption that movement takes place within a fractal space-time.

Thus, de Broglie's difficult problem can be solved, meaning that this could not justify the uniform movement of the particle within the wave field (incompatibility with the straight-line, uniform movement of the wave-corpuscle duality). The postulate through which motions are introduced on continuous and nondifferentiable curves solves this problem of the straight and uniform movement, meaning that on the new fractal manifold the movement is free (on geodesics). By accepting such a postulate, on the basis of the model of Scale Relativity Theory, it results that the geodesics of a fractal space-time supports a double representation, a stochastic, unpredictable one, described by Schrödinger type equations and specific to the wave character, and at the same time a deterministic, predictable representation, through the fractal hydrodynamic model, which is specific to the corpuscular character.

In Schrödinger's representation, only the modulus of the square wave function has physical significance, while in the second case we talk about average movements of some fluid particles which are submitted to a datum force, a force which is induced by the unpredictable part (non-differentiability of the motion curves). Non-predictibility, described through the non-differentiability of motion curves can be related to a Shannon-type fractal informational entropy, which, based on a maximization principle, leads to an egalitarian uncertainty principle. Within this uncertainty principle, the interaction constants are specified on the basis of an Onicescu-type informational energy. Now, we mention the fact that only the constant value of the Onicescu informational energy settles the interaction constants within the uncertainty relations.

Through the maximization principle, the integrally invariant functions are simultaneously probability density and movements on constant energy curves. Practically speaking, through the principle of informational maximization, the unpredictable, wave character given by the probability density is linked to the corpuscle character given by the energy.

The unpredictable part must be directly correlated to non-differentiability and is manifested through the existence of a potential, also called fractal potential. The principle of maximization of the informational energy gives a concrete form to the potential and the latter, introduced in the fractal potential, gives complete form to the force field. As a result, the informational energy not only stores and transmits the information through interaction, but also connects it directly to the deterministic part through interaction. So, practically speaking, the owner of all ”mysteries” is the fractal potential, which imposes the intelligent, fractal medium and the informational energy which gives the force.

As above-specified, on the basis of the non-predictable component, one can define a fractal entropy in Shannon's sense and, starting from here, a fractal informational energy in the sense of Onicescu. By using a maximization principle of fractal entropy in Shannon's sense, one can demonstrate that, if fractal informational energy in Onicescu' sense is constant, then the ratio between the corpuscle energy and the frequency of the associated wave is a constant at any resolution scale. As a result, the wave-corpuscle duality is achieved through movements on curves of informational energy constant in Onicescu's approach (for details, see [1-7, 11-13]).

4. Information as an expression of topological transformations. Different levels of reality

Topology studies the deformations of the space through continuous transformation, practically-speaking the properties of sets which remain unchanged at some transformations. Movement is a fundamental aspect of the real world and any elaborate study of dynamics leads to topology, as long as there is a dimensional space. Nevertheless, applications of the topological ideas appear in various fields, such as the theory of chaos, the quantum theory of fields, molecular biology, where the description and analysis of twists and deformations of the DNA molecule needs topological concepts. More specifically, the so-called topology of the knots allows for understanding the way in which the two spiral chains which make the double elicoidal structure of the DNA molecule can be unfolded when the genetic plan controls the development of the living being.

Starting from quantum microcosm towards our Newtonian reality, we meet the information under the same topological forms at every scale. Atoms form molecules and macromolecules, whose spatial configuration suffers topological modifications which grant them some properties. Organic macromolecules in protein and enzyme form ‘’ship’’ the information to cellular receptors, under the form of topological structures. Any modified radical determines a reconfiguration of spatial structures, which generates a certain property necessary in the chain of metabolical transformations which in this way are topologically equivalent, as they are obtained through topological transformations.

Any biochemical structure represents a graph, every cellular structure represents a network which forms knots and whose dynamics can be described by the network topology, which explicitly mentions the vicinities of every point. All this information comes from the structure of the DNA. The latter, apart from the succession of nitrate bases which form the genes, has a topologically-complex structure, in agglomerations which form the chromosomes, but which also influence the coding functions. The same information transmission mechanisms from DNA to RNA messenger and RNA ribosome and the constituting of protein and neurotransmitters we can also find within the structuring and functioning of the nervous system. We meet networks, knots, graphs, thus topological transformations also in this instance. All these represent only one part of the reality, because atoms, molecules, macromolecules, etc., are bodily aspects of the wave-corpuscle duality. All these structures have also a wave part, they are practically doubled by a spectral reality, of electromagnetic field.

The term topology is used also for establishing the projecting manner of a network. In order to highlight the physical (real) and logical (virtual) inter-connections between the knots, one can distinguish two corresponding types of topologies: a physical and a logical one, respectively. The physical topology of the network refers to the configuration of the transmission environments, of computers and peripheral devices, whereas the logical topology represents the method used to transfer information from one computer to another. The theory of domains developped within lattices represents a modality of modelling the topological concepts in a computational form, which allows for the processing of information.

Now, coming back to the wave-corpuscle problem, an analysis of the particle behaviour can be made from the perspective of fractal space-time, with the unpredictable and non-linear evolution, allowing that, on the basis of the informational theory of Shannon, we connect it to entropy and further, through a maximizing process, to the informational energy in the acception of Onicescu. There still remains an essential question: where can we search for and find the information in this quantum dynamics. It must be present both in the wave structure and in the particle properties. This connection cannot be made otherwise than in the phasic component of the wave, which is to be found in the spinning of the particle and which allows for the transfer of information from the spectral reality to the corpuscular reality, as it is demonstrated by the transform and the reverse of the Fourier transform. The phase is given by the magnetic component of the electromagnetic field and it represents the unpredictable, potential part, described by the complex function of Schrödinger’s wave formula, as these characteristics can be explained both through the fractal theory and through the topological transformations supported by the phase from the electromagnetic wave, respectively by the spin from the particle description.

The spinning movement is mathematically modelled using the complex analysis. This model is dynamic, as it undergoes transformations at the level of topological dimensions through the successive passage from the topological dimension 0 (of the point) to the topological dimension 1 (of the line) etc. Thus, a complex, infinitely-dimensional space is made, which explains the difficulty of highlighting the informational component. The successive passage through Euclidean, fractal and topological dimensions determines a quantitative, but also qualitative dynamics of energy. The moment in which this qualitative diversity is expressed is given by the moment of topological transformations at every dimension. This diversity which is practically unlimited renders quality, apart from quantity, to energy in its dynamics. From the perspective of complex systems we can find, in the statements above, the main characteristics specific to complex systems: non-linear dynamics, fractal geometry, with a latent informational energy which is potential, along with a dynamics of a practically infinite diversity, obtained by topological transformations in the phase complex space.

If we accept that topological transformations are invariant as compared to the scale and that these topological transformations represent energy patterns, configurations through which information is expressed, it should happen that, irrespective of the level of reality and of scale, the information has as an underlayer these topological transformations. The consequence of this is the ubiquity of information, just as the substance and energy, both at the level of microcosm and at macrocosm level.

Obviously, there exists structural information which, along with energy and substance, structures the matter at different scales and aggregation states. It is a structural information, which is achieved through topological transformations in fractal dynamics and even in euclidean dynamics. The topological space represents the place where information gains diversity, whereas energy gains a qualitative character. Qualitative variations of energy appear here, which constitute the informational energy or the psychological energy at mental level. Jung, in his research [16] over the unconscious and archetypes considers psychological energy to be a form of energy described through qualitative, not through quantitative ones, as physical energy was described. We will detail these considerations further below.

5. Dynamics of the real space – complex space in the structure of reality and psychism

Complex functions mathematically describe physical phenomena which assume the rotation movement around the own centre, including the movement of the magnetic vector of the electromagnetic wave, as well as from the fluid dynamics, and they sustain such hypotheses, theories and phenomena that the modern technology presupposes. This phenomenon is present first of all in the electromagnetic waves and thus it can be found in many situations which are theoretically and technologically described. The electric field corresponds to the real part, whereas the magnetic field corresponds to the imaginary component. The magnetic vector has a spinning movement, which is described by complex functions. At a 90-degree rotation (multiplication by i), an inversion of the components of the complex number takes place, a movement which in physics implies a Wick rotation. By multiplication with i , the amplitude and the phase are mutually modulated and their correlation is achieved by information.

The unpredictable, a-causal, unstructured, potential part of the complex systems structure can be found in the structure of the spectral field, associated to the corpuscle from the structured, causal, Newtonian, predictable part. This spectral component contains, through the imaginary component of the wave formula that describes the phase (the dynamics of the magnetic vector) the access to complex spaces, where the whole information can be found, structured in the topological geometry of energy configurations. The infinitely-dimensional possibility of these complex spaces, just as the infinite diversity of topological transformation within these spaces, along with their scale invariance allows for the estimation that in this infinitely-dimensional complex space we can have access to the whole information of the Universe. Thus, in a unitary approach, one can find the theory of complex systems, which comes from a physical perspective of the fluid physics, fractal theory, chaos theory and topology, with the complex analysis and the complex functions which use complex numbers with their imaginary component and which describe, in physics, the imaginary, unpredictable, potential, non-differentiable part, which can be found in the theory of complex systems.

As in mathematics information can be stored or processed by algebrical equations or by trigonometrical functions, in physical reality also information can be either algebrically or geometrically structured. The Fourier series and the Fourier transform achieve this through the interface between a spatial-temporal reality and a spectral one. Because the spectral reality is a-temporal, a-spatial, the Fourier transform and the reverse of the Fourier transform make this switch between the algebrical and the geometrical description. The mathematical model for the complex spaces includes the existence of topological transformations within an infinitely-dimensional space. As a result, the reality of the wave formula as being a-temporal, a-spatial, represents an interface between the newtonian reality and the complex ’’reality’’ of the complex spaces (Hilbert space).

The discontinuity of reality described by Planck as an energy quanta, by Gabor as information quanta, the non-differentiability specific to fractal dynamics, as well as the property of complex systems with deterministic chaos, all are due to a continuous interference between the physical and the complex reality through the spectral field. Depending on the local field conditions, of forces and scale structure, with the action of attractors, information from the complex space is absorbed (qualitative energy patterns, diversified through topological transformations), in order to structure the quantum or comsic Euclidean space.

The dynamics between the complex and the physical space is an expression of the mathematical description of reality through algebrical or trigonometric equations. The potentiality can be encompassed, codified in trigonometric equations and it expresses the information in an a-spatial, a-temporal reality which is specific to the wave and which is algebrically transformed in geometric form when a spatio-temporal reality appears, as it happens when the wave collapses into a corspuscle. In both cases, topological transformations are possible (in an a-spatial, a-temporal situation trigonometrically expressed, but also in a spatial-temporal one, expressed algebrically or geometrically).

Another argument of the physical character of the complex space is the wave function and wave function equation description which impose the existence of the Hilbert space. This abstract space allows for the description of the wave function and of the Schrödinger wave function equation. This space imposes the inclusion of both the real part of the wave and of its imaginary, complex one. As a result, the Hilbert space has properties of the complex space (the infinitely-dimensional character), the description by complex functions (complex analysis), but also the real part which includes the amplitude of the wave and its potential capacity of becoming real in the collapse of the wave formula. Another element which belongs to the real part is the space-time continuum character which we can find in the Minkowski space, but which we also find concentrated in the Hilbert space in the characterizing expression of the wave as being ,''a-spatial", ''a-temporal". In our view, the Hilbert space is an interface between the real space and the complex space and a proof that the complex space is a physical space connected through a permanent dynamics with the real space, as long as we accept the wave as real, with its wave function and wave equation.

The dynamics between the complex and the real space (the neuronal network), by way of the spectral field (wave field represented by the totality of the waves associated to the corpuscles in the neuronal network) is the basis of the psychological system functioning. This paradigm can generate new hypotheses which should explain the mysteries of the psychological life, just as the old ''mind-brain" duality. This new topic structure of psychism, associated with the theory of complexity and simplicity, applied to fractal geometry, through which reality is structured, allows the brain to have access also to the knowledge of the fractal as a whole, when the mathematical model is reduced as a number of informational bytes, to put it different as a symbol, but also, through the analysis and synthesis capacity, to be able to conceptualize the fractal at any point or at any scale, with the cost of an enormous informational content.

From a physical viewpoint, at any scale, there is a differentiable hydrodynamic description mathematically modelled by hydrodynamic equations, but also a stochastic, potential description, expressed by the equation of the wave formula. If we accept that the Hilbert space presents both the properties of the Minkowski space and the ones of the Euclidean space, just as of the infinitely-dimensional complex space, then it results that the Hilbert space represents the interface between the real space with all its descriptions and the complex space with all its potentiality.

Thus, the whole psychological life can be considered to take place in this Hilbert space which allows also for a Minkowskian perspective, a spatial-temporal continuum, under the form of the fractal space-time, where the information trigonometrically stored is a-spatial and a-temporal, thus creating the conditions of a stable memory, but also a spatial-temporal tri-dimensional perspective which represents sections in time and space of this continuum. This material component of the neuronal network allows for the processing of information, but for the superior psychological processes, the processing is achieved in the complex space, so that the synthesis, generalization, abstractization, conceptualization, all assume a multidimensional perspective, which can be made only in the infinitely dimensional complex space. More precisely, the dimensional dynamics from the 0 dimension to infinitely dimensional which in our reality is realized only up to three dimensions, can be realized multidimensionally in the psychological reality in the complex space (through the fractal potential).

6. An approach from the perspective of the complex systems theory for the processing, storage and transmission of information at brain level

As we already know, a complex system cannot be analyzed on principle through the part fragmenting, as it is made up of elements which make sense only within the privacy of the system. It has an unpredictable evolution (than, mostly, within a short time frame called temporal horizon), can suffer sudden transformations, no matter how big, without obvious external causes and it manifests different aspects according to the analysis scale. It is on principle different from a complicated system because the difficulty of prediction is not to be found in the inability of the observer to analyze all the variables which would influence its dynamics, but in the sensitivity of the system to initial conditions (slightly different initial conditions which lead to extremely different evolution possibilities), to which one can add the effect of an auto-organization process (process determined by the very interactions between the component sub-systems and whose effect is the spontaneous emergence – principled unpredictable – of some order relations).

A complex system can be modeled and studied within a topologically-equivalent space, called the phase space, in which specific notions are defined: attractors and repulsors, attraction basin, trajectories, limit cycles, etc. In this context, one can talk about a functional modeling, which is much more abstract and ''unbound'' from the constraints imposed by a concrete ''anatomy'' and ''physiology''. While classical modeling starts by approximating what ''can be seen'', functional modeling involves the identification of an equivalent dynamic system, whose behavior can be analyzed through specific methods with an extremely high degree of generalization.

In the systems composed by a great number of elements, the properties of the systems cannot be found in the total amount of the complex systems properties. The emergence property is what creates a link between the multitude of the components and the properties of the complex systems.

All these theories are trying to describe, as close to the reality as possible, the privacy of the functioning of systems with a great number of elements, which interacts with other systems (dissipative systems) and which in fact are widely-met in the physical reality. These systems have a series of properties, among which the emergence is one with special implications, but also the dynamic structure they presuppose, generally characterized by a structured, causal, Newtonian, predictable component and an unpredictable, a-causal, unstructured, potential one. Physical experiments (the ones in the plasma tubes but also in the fluid dynamics, etc.) have highlighted these components just as the dynamics between them, which presuppose an auto-structuring tendency by means of the attractors within a certain space called the phase speace. However, there remains an important question connected to the source of information which allows for the auto-structuring and thus the dynamics between the potential component and the predictable one. In the plasma tubes experiments, the phenomena can be more easily observed because upon modification of the system constraints (modification of electrical tension to the two ends of the tube) we can obtain different particle organisation patterns which presuppose the interference of some informational structures. The current explanations for the source of this information is that it comes from the privacy of the system. However, in the structure of the system (if we stay with the more simple model of the plasma tube) there are only particles and their attached wave component. Considering that the information contained by the particle comes from the coherent wave, the question which arises is where the information comes from, at wave level. In everyday life, today, in the information technology era, the information is trasmitted via waves, by their analogical transformation into waves which modulate a carrying wave. Modulation can be the modulation of the amplitude (little employed because it is too easily affected by noise, but anyway the amplitude is in inverse ratio to frequency), the generally-employed modality is that of angle modulation, which presupposes a modulation of either the frequency or of the phase, which is finally transmitted to the modulation of the magnetic vector angle. The phase is recognized as being an imaginary, complex component of the wave formula. The movement of the magnetic wave described by complex equations generate a complex plan, which connects the wave to the complex space and allows for the storage of information in the topological modifications from this infinitely-dimensional space. To put it different, the information in the complex systems is to be found in the complex space, which gives the characteristics of potentiality, non-differentiability, a-causality from the description of complex systems.Coming back to the plasma tubes, the privacy of the system from which information comes is represented by the coherent wave phase with every particle (the wave corresponding to every particle from the wave-particle duality) which represents the connection to the complex space, where the whole physical reality is to be found at the potential mode, under the form of information. This is the consequence of permanent dynamics between the complex and the real space, by means of information. Depending on the system constraints from the complex space through the wave phase, the information which reaches the particle generating the auto-structuring patterns is undertaken.

In the structure of complex systems there is a potential part with a chaotic aspect and a structured, causal, newtonian part, as well as different intermediary phases. From there it results that a certain uncertainty exists in all the structure of reality. Moreover, we find the uncertainty principle (Heisenberg [14]) in Gabor's theory of communication (the information quanta). At brain level, the non-linear, potential, apparently chaotic part corresponds to the unconscious, whereas the structured, causal part corresponds to the conscious; the intermediary parts, as well as the structures which process both the information from reality and from the unconscious, all are represented by what Freud called SuperEgo.

The chaotic part is structured via attractors, depending on the constraints of the system (for example, the way in which some physiological needs generate, during the dream, a certain structure). During the wakefulness there is a dynamics with the chaotic, potentially unconscious part in the background and which allows accessing the information, the memories, the logical links (for example, a discourse).

We must therefore accept that, also in the living world, including the brain functioning, there exists a spectral, wave component and the transmission of senses is achieved spectrally, by vibrations [9, 10].

Thus, a reality which coexists with us is the a-spatial a-temporal one, described by the wave formula and which is involved in the visual perception phenomenon, in which the undertaking of the spatial-temporal information is made by light through modulation of its frequency, a phenomenon which is described by the Fourier transform, while the stimulation of the retina involves the collapse of the wave formula and the emergence of corpuscles which stimulate the retina cells by inversing the Fourier transform. As a result, all we look at and see, in order to be seen, goes through an a-temporal and a-spatial phase, in the interval necessary for the light to reach from the object to us. This interval can be million of light years for cosmic objects or minutely small fractions of a second when we look at our friends, our home or our garden.

Information is codified energy which is expressed as pattern, structure templates, innitiated by attractors which are active in the phase space, between the chaotic part and the structured one. The information lies stored in the spectral space and it expresses the patterns in the structure of atoms, molecules, macro-molecules and cells. It has a potential existence which is expressed by substance and energy under certain conditions (of local coherence).

A virtual, Newtonian reality as projection of physical reality is completed by the unstructured, a-causal, apparently chaotic component: the imagination, the dream, the failed acts, the subliminal mechanisms, the unconscious etc., which can be associated with the causal, potential, unstructured and non-differentiable component of complex systems, the source of inspiration, of creation and of access to non-Euclidean realities to holospace. These potentialities can become conscious through patterns (see the archetypes and the collective unconscious of Jung) and they can be found in logical, algorithmic, organized and systematic form in everything that is creation (from making a speech, conversation, improvisation, to creating new musical pieces, new artistic work, new scientific work). The chaotic, unpredictable part does not only contain the Newtonian reality to which we have access, but more, maybe even the structure of the whole Universe, at informational potential level. The brain has access to the implicit part (the implicit reality of Bohm [9]), if we associate this part to what the classics called unconscious. From here derives the capacity for mathematical reasoning, for physics, for reasoning reality in n dimensional spaces, a-temporal realities, a-spatial realities.

The development of the neuronal network is made according to fractal criteria, just as all the other apparatuses and systems of the human body. As a result, the spectral field formed by the waves corresponding to corpuscles form the neuronal network are coherent, allowing for the processing of information both in the neuronal network and in the spectral space (the Hilbert space), where both the a-spatial a-temporal components exist, as they allow memory to develop, but also the complex component which offers the possibility of a multi-dimensional processing which can explain the superior psychological processes (conceptualization, semantics, abstracting and generalization etc.). At any scale we can find the two types of realities: a differentiable one and a non-differentiable one, highlighted by the Madelung hydrodynamic model Madelung and respectively by the stochastic model.

Analyzers manage on principle the transformation of wave information in the corpuscle, thus generating the tri-dimensional and the spatial and temporal vision upon reality, but the processing at brain level is also spectrally made (de Valois [20]). Because the whole of the analyzers achieve the information transfer from a wave form to a body form, the processing of information is achieved both within a material, corpuscle network, the neural network, but also in a spectral network, of the coherent field associated to the neuron network. Through the waves of the spectral field the dynamic link to the complex space is made, process which allows for the occurrence of the superior psychological processes, specific to the human being, which need multidimensional development in order to be formed, development which is only allowed by the complex space. The psychological reality represents the permanent dynamics between the neuronal (material) network, the associated spectral field (the fractal potential) and the complex space (infinitely-dimensional).

The processing of information for the information provided by analyzers is made in a differentiable, causal, algorithmical form at the level of the neuronal network (Barabassy [8]), whereas beyond the analyzers it is made within a complementary network found in the complex space, which is mediated by the fractal potential from the spectral potential of neurons. As a result, the qualitative leap represented by the appearance of culture would not have generated only the appearance of mirror neurons which are present also in some animals, but the development of genetic patterns which allowed for a better connectivity between the two networks. It is not a matter of chance that the appearance of articulate speech is associated to this qualitative leap in human development. The centre of speech appears to represent a system of processing information which allows for connecting to the infinite and complex-dimensional space and thus to the possibility of emergence of superior psychological processes. The study of the mimic-gesture language of the deaf individuals highlighted the fact that, in learning this language, there is a first phase of learning of a mimic-gesture behaviour which is processed in the right hemisphere, which is specialized in spatial representations and becomes truely language only when it is undertaken by the speech centre from the left hemisphere. Then the mimic-gesture is processed at the level of notions and concepts and superior processes can be achieved, such as abstractization, synthesis and generalization. The result is that the centre of speech can be such a module which allows for the connection of the neuronal network with the one corresponding in the complex system and thus the leap towards Homo sapiens can be explained.

Concluding remarks

The complex systems dynamics and especially that of the complex and of the real space (from the inner part of the systems) may lead to new hypotheses and theories about the structure of psyche and about its functioning.

The whole collection of the analyzers manages the transfer of information from its wave form into corpuscular form. This allows for the information processing to be accomplished both in a corpuscular, material network, the neuronal network, but also in a spectral network, of the coherent field associated to the neuronal network. Through the waves of the spectral field the dynamical link to the complex space is realized, situation which allows for the occurrence of the superior psychic processes, which are specific to the human being, and which need multidimensional development in order to be formed, a situation which is only allowed by the complex space. The mental reality represents thus the permanent dynamics between the neuronal (material) network, the associated spectral field (the fractal potential) and the infinite dimensional complex space.

The aim of this paper is to apply the theory of complex systems in order to sustain the hypothesis of the complex space as a physical space. We want to treat the information correlated to the substance and the energy, by applying the theory of complex systems, of complex analysis and of topology. We aim to highlight the fact that information can be found in the complex space of the wave phase spectral field. As a result, this complex space can be found anywhere and at every level of the reality. In our view, it is infinite dimensional, as it can contain all the information in the Universe. From a mathematical viewpoint, the real space is included in and intertwined with the complex space generated by the electromagnetic waves. At quantum level, this intertwining can be achieved by the collapsing of the wave formula into the complex space of the wave phase and it can be transmitted into the complex space of the spin rotation, by transferring the whole information. This phenomenon is specific to reality at the level of the whole knowable universe, as everywhere there are electromagnetic waves and also at every level of the reality, including the human brain.

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