The Acoustics of Bells

Thomas D. Rossing

Studying the vibrations of large and small bells helps us understand the sounds of one of the world's oldest musical instruments

Bells are probably older than recorded history. One of the oldest that still survives, found near Babylon, is thought to be about 3,000 years old. A set of tuned bells dating from the fifth century B.C. was recently discovered in the Chinese province of Hubei. Bells developed as Western musical instruments, however, only in the fifteenth and sixteenth centuries, when bell founders discovered how to tune them harmonically. The founders in the Low Countries, especially the Hemony brothers, took the lead in tuning bells, and their many fine carillons became a source of national pride, which they still are.

Analysis of the rich sound of a bell reveals many components, or *partials*, each with its own individual decay time, as shown in Figure 1. The most prominent partials in the sounds of tuned church bells or carillon bells, like those of most musical instruments, are *harmonics* of a fundamental (that is, their frequencies are multiples of a fundamental frequency). But in addition to the harmonic partials, bell sounds abound with inharmonic partials as well.

Bell founders normally tune the first five partials to have frequencies in the ratios 1 : 2 : 2.4 : 3 : 4. This tuning results mainly from careful design of the shape of the bell, but the fine tuning is done by removing metal from the inner surface of the bell, as will be described later.

The acoustical properties of bells received Lord Rayleigh's attention in the latter part of the nineteenth century (1890, 1894). He studied the bells in the tower of his own parish church at Terling and also performed experiments on several bells in his laboratory. He was able to identify four to six partials in each bell, and he correctly concluded that the nominal pitch of each bell—the note that most listeners hear—was determined by the fifth partial (labeled "octave" in Fig. 1).

The pioneer investigator of bell acoustics in the United States was A. T. Jones, who studied the bells of the Carlile Chime at Smith College and the Memorial Carillon in Harkness Tower at Yale University (1928). By carefully examining each bell with a stethoscope connected to a Helmholtz resonator, Jones mapped the nodal lines for the first seven partials, and he also investigated the origin of the strike note, the subjective tone chiefly responsible for the pitch of a bell.

Since World War II, extensive investigations of the acoustics of church bells and carillons have taken place, mainly in the Netherlands, Germany, and England (see Rossing 1984 for references). Especially noteworthy is the work of van Heuven (1949), who reports the results obtained during his extensive studies of bell acoustics for the Department of Preservation of Arts and Monuments in the Netherlands.

Thomas Rossing is a professor of physics at Northern Illinois University. His research interests include musical acoustics, environmental noise control, and the physics of solids. He is the author of over 100 publications, including six books, and holds eight US and eleven foreign patents. He plays several musical instruments and has directed church choirs and handbell choirs. His love of bells began with his introduction to change ringing on the historic bells of Old North Church in Boston. Address: Department of Physics, Northern Illinois University, DeKalb, IL 60115.

Vibrating plates and Chladni's law

When struck by its clapper, a bell vibrates in a complex way. Many elements of that vibration bear a resemblance to those of a flat circular plate, illustrated in Figure 2. As the drawings show, the vibrating plate develops regions that remain relatively stationary, called nodes, and regions where the back-and-forth motion is most intense, called antinodes. Each particular configuration of nodes and antinodes is associated with a mode of vibration.

On a circular plate, nodal lines can form in many ways, including circles and diameters. The many modes of vibration are often described by two numbers (m,n), which give the number of nodal diameters and circles, respectively.

Nearly 200 years ago, E. F. F. Chladni described some well-known experiments on vibrating plates (1787). By sprinkling sand on their surface, he was able to see where the nodal lines formed, because the sand bounced away from the antinodes and settled on the nodes. Chladni observed that the addition of one nodal circle raises the vibrational frequency of a flat circular plate by about the same amount as adding two nodal diameters, a relationship that is sometimes referred to as Chladni's law. Thus the frequency of a particular mode of vibration can be written as $f = c(m + 2n)^p$, where m and n are the numbers of nodal diameters and circles, and c and p are appropriate constants.

The vibrational frequencies measured in a wide variety of flat and nonflat circular plates can be fitted to this empirical law of Chladni. In flat plates, p is nearly 2, but in cymbals, bells, and gongs it may vary from about 1.4 to 2.4 (Rossing 1982). The observed frequencies in a modern English church bell, for example, can be fitted to Chladni's law using values of p ranging from about 2.0 to 2.3 for different families of modes (Perrin et al. 1983).

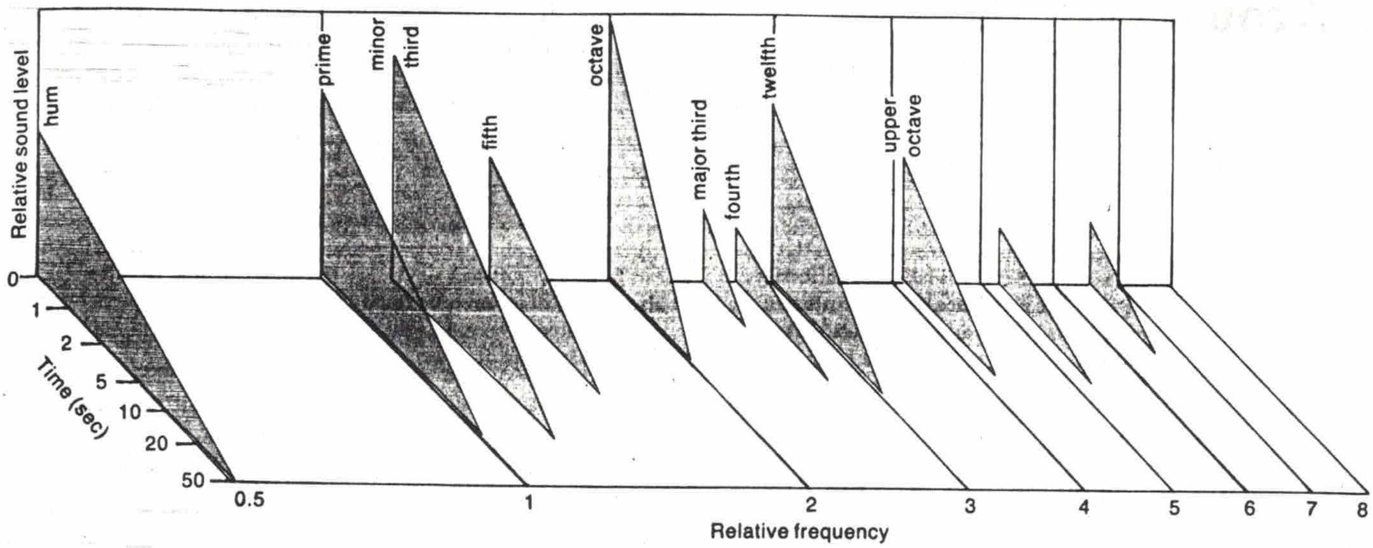

Figure 1. The rich, musical sound of a bell is composed of a spectrum of partials, each with its own frequency and decay time. In tuned bells, many of the strongest partials are harmonics, or integral multiples of a fundamental frequency, as shown here. Each doubling of frequency corresponds to a musical octave; a few partials fall near musical thirds or fifths. Each partial is produced by a specific mode of vibration (see Figs. 2 and 3) and is radiated in a different pattern. Thus, the tonal character of the sound changes with the location of the listener as well as with time elapsed after the bell has been struck.

Modes of vibration

In principle, the complex vibration of a bell can be described in terms of the normal modes of vibration. The most important modes are flexural, but to a lesser degree extensional, torsional, and thickness-shear modes also exist. Mode mixing, doublet splitting, and other effects complicate the vibrational spectra, especially at high frequencies.

Nearly all the significant partials in the bell sound result from vibrational modes in which the motion is mainly in the axial plane; that is, the motion is primarily bending. It has been customary to classify these modes into families, or groups with some common property. The most important families are those whose members have antinodes where the clapper strikes. In a church bell or carillon bell, this region of maximum vibration occurs at the sound bow, whereas in a handbell the antinodes develop a short distance above the mouth.

The first five modes of a church bell or carillon bell are shown in Figure 3. The (2,0) is the only mode without a nodal circle. Note that there are two modes with m = 3 and n = 1, the first with the circular node at the waist and the second with the circular node near the sound bow. Thus, we follow the custom of Tyzzer (1930) and others and denote the latter as (3,1#).

A detailed study of an English church bell has compared the normal modes computed by a finite-element method to the first 134 modes observed in the laboratory (Perrin et al. 1983). Modes such as (2,0), (3,1), and (4,1) are classified as "ring driven," since an antinode occurs in the vicinity of the sound bow. They are strongly excited by the clapper, and they radiate most of the strong partials in a bell's sound. Bell founders often refer to them as group I modes (Lehr 1965). The second important family, designated as group II, includes the (2,1#), (3,1#), (4,1#), and higher modes. They are classified as "shell driven," because the circular antinode occurs near the waist as a result of the entire bell's vibration, rather than being directly excited by the clapper.

Other important families of normal modes are those with n = 2, 3, 4, . . . , referred to by some founders as groups III, IV, V, etc. The relationship of modes within a single group can be seen graphically in Figure 4. Modes in group III have a nodal circle near the waist and also near the sound bow. The second mode has a nodal circle about halfway between the sound bow and the waist, and so in one sense it could be considered to be a combination of the (2,1) and (2,1#) modes.

Modes in a handbell

Tuned handbells, which developed in England during the eighteenth century (Price 1973), have some properties in common with church bells but are also different in several ways. They are much lighter in weight than church bells, they do not have a thick sound bow, and they ordinarily have only two tuned partials. Their nominal pitch is that of the lowest partial.

Handbells have some modes of vibration that do not exist in church bells—(3,0), for example, and sometimes (4,0) in larger bells—and the order in frequency of the various modes depends on the size and shape of the bell. The (2,0) and (3,0) modes always produce the lowest-frequency vibrations. The next highest in frequency may be the (3,1), (4,0), or (4,1) mode, however, depending on the size of the bell (Rossing and Sathoff 1980).

The modes of a handbell, like those of a church bell, can be arranged into families or groups according to the number and position of nodal circles. In the laboratory, we have used several methods to study them. It is difficult to isolate single modes when the handbell is struck by its clapper; they combine in complicated patterns. Therefore a tiny permanent magnet is attached to the bell and is driven electromagnetically at a single frequency with a solenoid and audio amplifier. Frequencies corresponding to the normal modes of vibration can be

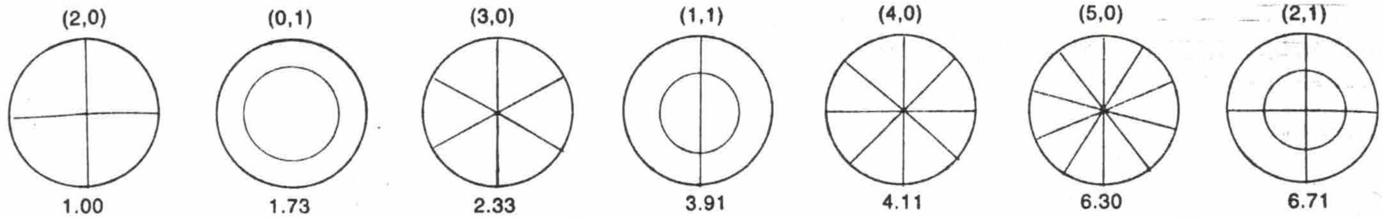

Figure 2. A vibrating surface develops standing wave patterns at resonant frequencies: it has lines that are relatively stationary, called nodes, and areas where the movement is most intense, called antinodes. The patterns of nodes and antinodes that develop in a vibrating bell are similar to those that form in a flat circular plate. This drawing illustrates the first seven modes of vibration in a circular plate with a free edge. Nodal lines (*color*) can form as diameters or circles; the numbers in parentheses above each diagram give the numbers of nodal diameters and circles, respectively. The relative frequencies of each mode are given below the diagram.

identified by probing the sound field very near the bell with a small microphone. However, it is difficult to locate the nodal lines exactly by this method, especially in a small bell. Therefore, in collaboration with Richard Peterson at Bethel College, we have used time-average hologram interferometry as a means of analysis.

A hologram is normally recorded on photographic film by allowing reflected laser light to interfere with a reference beam from the same laser. Such a hologram contains information from which the original image can be reconstructed when reilluminated by laser light. When the object is vibrating, however, the reflected light constantly changes, and interference patterns develop in the reconstructed image which contain a considerable amount of information. The nodes are seen as bright lines, and interference fringes map the relative amplitude of motion in the antinodal region (Powell and Stetson 1965).

Several hologram interferograms of a handbell with a C_5 nominal pitch are reproduced on the cover: from left to right, the top row shows the (5,2), (4,2), and (3,2) modes; the middle row (2,0), (6,1#), and (7,1); and the bottom row (2,1), (5,1#), and (5,1). These photographs offer striking visual confirmation of the many modes in which a handbell vibrates. We have noted, for example, that the first nodal circle is about one-fourth of the way up from the mouth in the (2,1) and (3,1) modes but moves up to the waist in the (4,1) and other modes. A theoretical study of this bell by means of a finite-element method confirms this, and complements the visual evidence in the holograms (Perrin et al. 1984).

Casting and tuning

Most bells of high quality are cast of bronze with about 80% copper and 20% tin. Steel bells are considerably less expensive and mechanically stronger, but their tonal characteristics are generally considered inferior. Steel or iron is widely used for clappers in church bells and carillon bells, however. The effects, good and bad, of adding small amounts of other metals to bell bronze have been discussed by Schad and Warlimont (1973) and by Hanson et al. (1976).

The methods used to cast large bells have changed very little through the years. The cope, or outer mold, is fitted over the core, or inner mold, both of which are built up with layers of loam, a mixture of sand and straw. In earlier years, the mold was buried in a pit to prevent it from bursting when filled with molten bronze. Nowadays most casting is done above ground, and the outer mold is built up inside a set of heavy iron rings. The pouring of molten bronze into such a mold is shown in Figure 5.

Although in theory it might be possible to cast a bell in the exact shape and thickness to ring with the desired harmonic partials, in practice it would be difficult to do so. It is more practical to cast a bell slightly thicker than necessary and turn it down to the correct thickness on a bell lathe, as the bell tuner is doing in Figure 6. This practice has been followed since the seventeenth century.

Because of the different vibrational pattern for each partial, a thinning of the bell at any particular place will

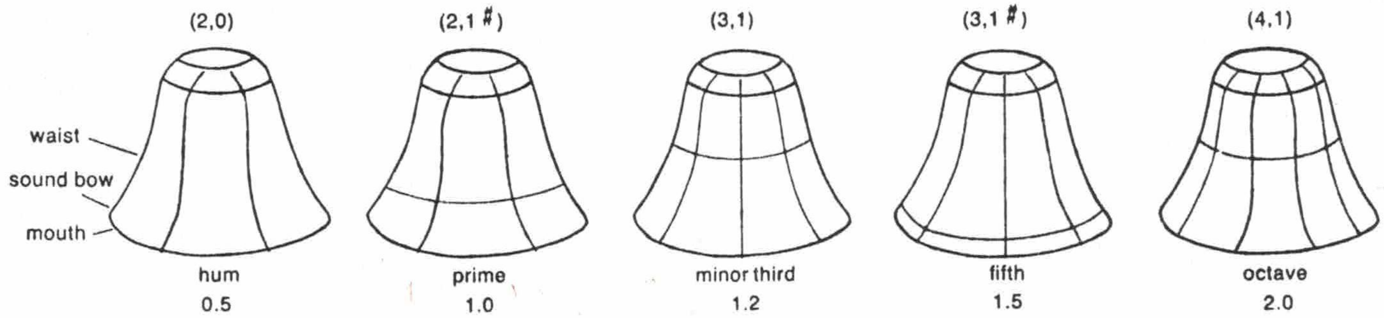

Figure 3. The nodal lines in a vibrating bell are analogous to the diameters and circles of Figure 2, and the same pair of numbers, the first for nodal meridians and the second for nodal circles, designates each mode of vibration. The first five modes of a tuned church bell or carillon bell are shown here; these modes of vibration produce the first five partials shown in Figure 1. The relative frequencies and names of these partials, which vary among bell founders of different countries, are given below each diagram.

change the frequency of the various partials by different amounts. This means that the bell tuner must know exactly what frequency changes will take place for thinning at each location. In order to know this, the tuner makes use of tuning curves, such as the set shown in Figure 7, which guide him in removing metal from the inner surface of the bell. A similar set of curves can be drawn for the outer surface (van Heuven 1949), but tuning is nearly always done on the inner surface because the outside is usually adorned with ornamentation and inscription.

In a well-tuned carillon, not only must the partials within each bell be tuned, the bells must be accurately tuned with respect to one another as well. Techniques for accomplishing this were developed as early as the fifteenth century, although it was during the seventeenth century that the Dutch founders Jacob van Eijck, François and Pieter Hemony, and others developed bell tuning and carillon building into a fine art.

Handbells require somewhat different methods of manufacture. They are frequently cast considerably thicker than the finished bell, and metal is removed from both the inner and outer surfaces in tuning. Most handbell manufacturers tune only the first two partials, and so tuning curves less complicated than those used for church bells or carillon bells suffice. The tuning of the two lowest partials is no less exact than in carillon bells, however.

Tonal character of bells

Although founders usually tune only the lowest five partials in church bells and carillon bells, several additional partials end up being close to harmonics of the lowest partials or at least close to the notes of a musical scale based on the bell's prime, or fundamental. Thus these partials are often referred to by the nearest note or interval on a musical scale.

Table 1 gives the names and relative frequencies of 12 prominent partials in the sound of a typical bell. The frequency ratios termed "ideal" are those of the corresponding notes on the scale of just intonation. Note, however, that the highest four partials in an actual D_4 church bell, given in the next to last column, are raised as much as 4% above the ideal frequencies. This "stretching" of the partial series may well contribute a desirable richness to the sound (Slaymaker 1970).

When a large church bell or carillon bell is struck by its clapper, one first hears the sharp clang of metal on metal. This atonal sound includes many inharmonic partials that die out quickly, giving way to a strike note, or strike tone, that is dominated by the prominent partials of the bell. Most observers identify the metallic strike note as having a pitch at or near the frequency of the second partial (the fundamental, or prime), but to others the pitch seems an octave higher. Finally, as the sound of the bell ebbs, the slowly decaying hum tone, an octave below the prime, lingers on.

The strike note is of great interest to psychoacousticians, because it is a subjective tone created by three nearly harmonic partials. The octave, the twelfth, and the upper octave normally have frequencies nearly in the ratios 2:3:4. The ear assumes these to be partials of a missing fundamental, which it hears as the strike note,

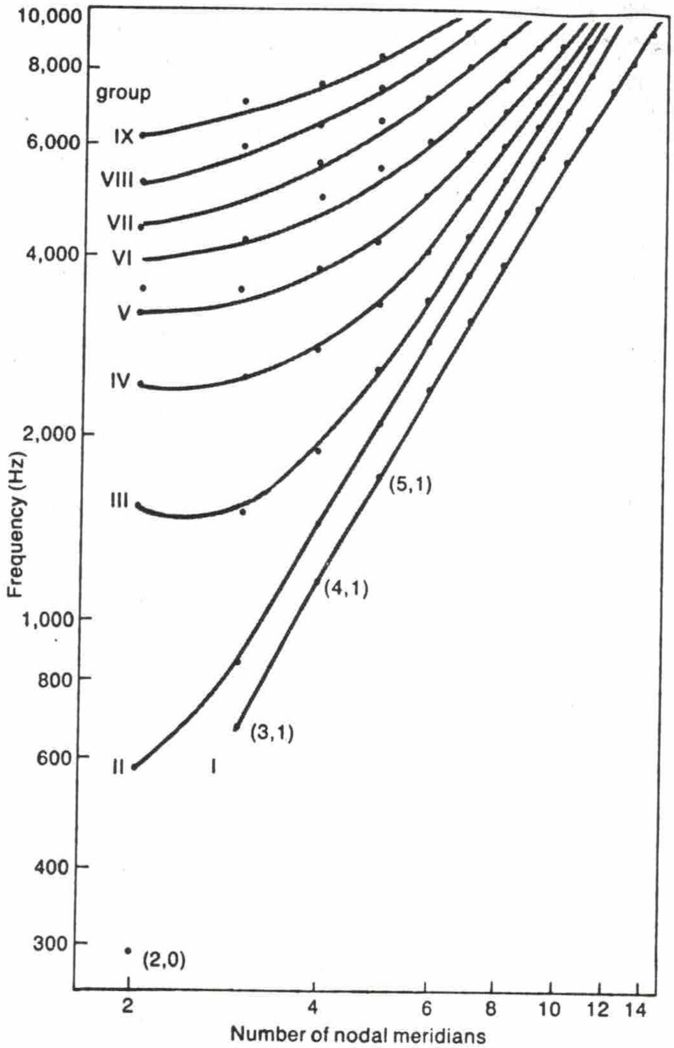

Figure 4. The various modes in a tuned bell are conveniently grouped into families, each family having the same number of nodal circles (in about the same location on the bell) and different numbers of nodal meridians. The vibrational frequencies of the modes in each group, when plotted logarithmically, lie along curves that approach straight lines as the number of meridians becomes large, and can be fitted to Chladni's law. The frequencies of this English church bell were measured by Perrin and co-workers (1983).

or perhaps we should say as the primary strike note.

In very large bells, a secondary strike note often occurs about a musical fourth above the primary strike note and may even appear louder under some conditions. This secondary strike note is also a subjective tone, created by four partials beginning with the upper octave. These partials are produced by the (6,1), (7,1), (8,1), and (9,1) modes of vibration, and they have frequencies nearly 3, 4, 5, and 6 times that of the secondary strike note. In a large bell of 800 kg or more, these partials lie below 3,000 Hz, where they may contribute a strong residue pitch (Ritsma 1967).

A handbell produces a much simpler sound; its fundamental pitch appears almost at the onset of the sound. There are several reasons for the absence of a separate strike note. First of all, there is no group of harmonic partials to create a strong subjective tone. Second, handbell clappers have surfaces of a soft, nonmetallic material such as leather or plastic.

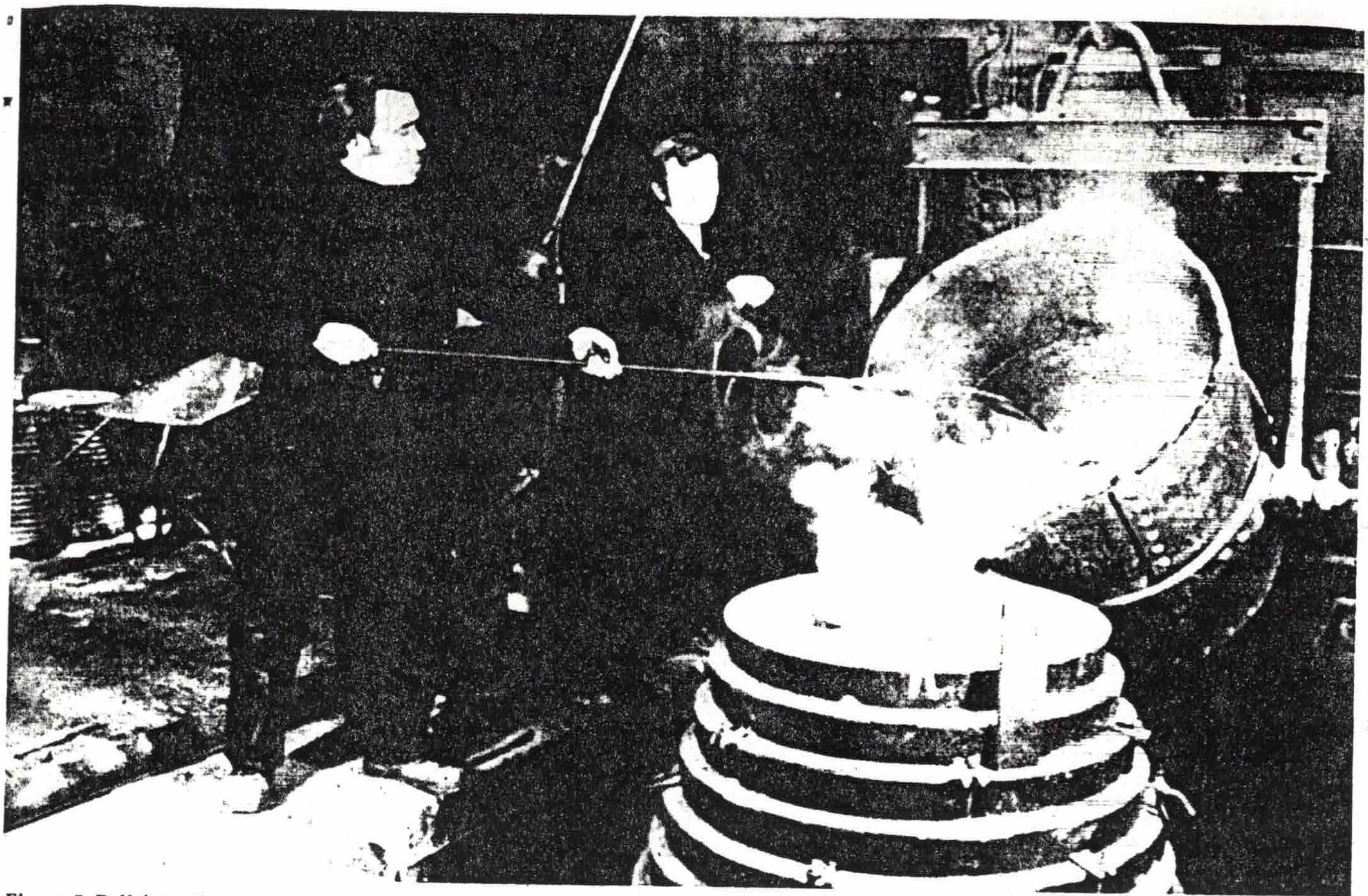

Figure 5. Bell founding has changed little over the years. In earlier centuries, the mold was buried in the ground to prevent it from bursting; nowadays the outer mold, or cope, is reinforced with iron rings. Molten bronze is still handled much the same way as in the fifteenth century, when the art of making bells started to flourish in the Low Countries. (Courtesy of the Dutch National Carillon Museum, Asten.)

Sound radiation, decay, and warble

The most prominent partials in the spectrum of a church bell or carillon bell are radiated by the ring-driven modes classified as belonging to group I. These modes are due to standing flexural waves in the vibrating metal, and they have a nodal ring about halfway up the bell. To understand the general properties of the sound-radiation field, we can model the bell's outside surface below the nodal ring as a collection of $2m$ sound sources alternating in phase, where m is the number of complete nodal meridians. (Figure 2 shows how the $(m,0)$ modes in a circular plate also form $2m$ segments; the adjacent wedges move in opposite directions, and every other one is moving the same way, or in phase, at a given instant.)

The radiation efficiency of such a collection of alternating sources increases rapidly with frequency and with the size of the bell. As the bell increases in size, the area of each source increases, and hence the radiation of sound is stronger. But an additional boost in radiation occurs when the physical separation of adjacent sources exceeds half a wavelength of sound in air. Another way to express this condition is that when the speed of flexural waves in the bell exceeds the speed of sound, radiation efficiency increases markedly. This is because air near the vibrating surface tends to flow back and forth between adjacent areas of opposite phase, creating a sort of pneumatic "short circuit" at vibrational speeds slower than the speed of soundwaves in air.

A quick calculation will give a sense of the variables involved. The speed of flexural waves in a plate is $(1.8c_L hf)^{1/2}$, where h is thickness, f is frequency, and c_L is the longitudinal wave velocity, which is equal to $(E/\rho)^{1/2}$; E is Young's modulus of elasticity, and ρ is density. For a church bell having a diameter of 70 cm, the flexural wave speed is roughly 320 m/s for the (2,0) mode, but it increases to about 645 m/s for the (4,1) mode and to about 1,200 m/s for the (9,1) mode. Since these are substantially greater than the speed of sound in air (about 334 m/s), the church bell radiates most of its partials quite efficiently.

In a handbell, the walls are much thinner, and so the flexural wave speed is considerably slower than in a heavy church bell. In the bell shown on the cover, for example, the flexural wave speed is roughly 100 m/s for the fundamental (2,0) mode. Since this is considerably less than the speed of sound, the radiation of this partial is not very efficient. The (3,0) and (4,1#) modes in the same handbell have flexural wave speeds of approximately 200 m/s and 300 m/s, respectively, nearer to the speed of sound; the corresponding partials tend to be radiated a little more efficiently than the fundamental. In large handbells, where the principal frequencies are all lower, the flexural wave velocity for all the principal modes is much less than the speed of sound in air. This is a significant problem in handbell design.

In addition to the direct radiation of sound normal to its vibrating surfaces, a bell also radiates sound axially at twice the frequency of each vibrational mode (Rossing sound increases with the fourth power of the vibrational amplitude, whereas the direct radiation increases only with the square of the amplitude.

The fundamental (2,0) mode in a handbell radiates a fairly strong second harmonic partial along the axis as well as a fundamental whose maximum intensity is perpendicular to the axis. The (3,0) mode also radiates at twice its vibrational frequency, but this partial is usually quite weak. The principal harmonic partials in the handbell sound are thus the first, second, and third harmonics.

A vibrating bell loses energy mainly by sound radiation, although internal losses also play a role. The sound pressure level of each radiated partial decays at nearly a constant rate (that is, the vibrational energy decays exponentially), and it is customary to express the decay time for each partial as the time it takes for the sound to diminish by 60 decibels. The decay times of the principal modes of the D_4 church bell in Figure 4 are given in the last column of Table 1. The long decay time of the (2,0) mode (hum) and the relatively short decay times of the modes of higher frequency are due mainly to the greater radiation efficiency of the higher modes.

Because of slight imperfections in the symmetry of many bells, a mode can sometimes split into doublets, or closely related vibrational frequencies, according to where the clapper strikes. The beating together of the nearly degenerate components of a mode doublet causes a "warble" in the sound of the bell (Perrin and Charnley 1973). In theory, warble can be eliminated by selecting the strike point of the clapper to lie at a node for one component and an antinode for the other, so that all the clapper's energy goes into one component and none into the other. But in practice this may not be possible, because doublet splitting is likely to occur in more than one mode, so that selecting a strike point to minimize warble in one mode may actually enhance it in a different mode. Other means of suppressing warble in church bells have been suggested, including the addition of two large meridian ribs to modify the circular symmetry of the bell (Perrin et al. 1982).

Warble is much easier to cope with in a handbell, because fewer modes of vibration contribute prominent partials to the sound. It is usually possible to select a strike point that reduces warble in both the (2,0) and (3,0) modes to a tolerable level. The few bells in which this is not possible are usually scrapped by the founder.

Scaling of bells

The fundamental frequency of a bell has roughly the same dependence on thickness, h, and diameter, d, as flexural vibrations in a circular plate: frequency is proportional to h/d^2. Thus it is possible to scale a set of bells by making all dimensions proportional; h/d remains constant, and frequency varies as the inverse of diameter. This method, known as $1/f$ scaling, approximates that found in many carillons dating from the fifteenth and sixteenth centuries. However, a $1/f$ scaling causes the small treble bells to have a rather weak sound, and later bell founders increased the sizes of their treble bells (Bigelow 1961).

To make heavier treble bells, they had to depart

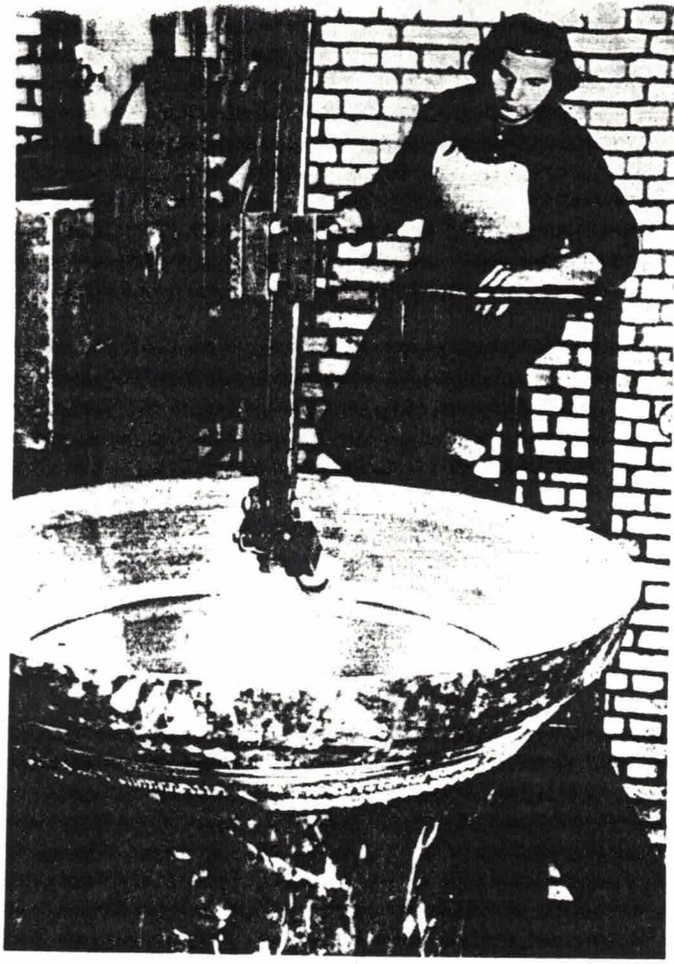

Figure 6. After a church bell or carillon bell has been cast, the first five partials must be tuned to a harmonic sequence. The tuner uses a bell lathe, and he consults tuning curves such as those shown in Figure 7 to decide where metal should be removed from the inner surface. (Courtesy of the Dutch National Carillon Museum, Asten.)

from $1/f$ scaling in the higher frequencies. Their method can be conveniently described by writing the frequency of a bell as $f = k/r$, where r is the radius. Whereas a set of church bells may have a constant k (that is, the bells keep constant proportions and follow a $1/f$ scale), in carillon bells k must increase with frequency so that the size of bells in the upper octaves is larger than it would be in $1/f$ scaling. The average value of k in several fine seventeenth-century Hemony carillons increases from 100 m/s in bells of 30 kg or larger to more than 150 m/s in treble bells. A study of a large number of carillon bells cast at the Eijsbouts foundry in the Netherlands shows that k increases from about 100 m/s for large bells (100 kg and upward) to about 200 m/s for the smallest bells (Lehr 1952). Since the mass of a bell increases as the cube of its radius, these treble bells are about eight times heavier than they would be if the foundry had followed a $1/f$ scaling.

Further studies by Lehr, as yet unpublished, have shown that the families of modal frequencies vary differently with the bell's thickness. In the modes of group I, frequency is roughly proportional to $h^{0.7}$. In the modes of group II, frequency is roughly proportional to $h^{0.86}$, which is nearer to the behavior (f proportional to h) found in flat plates and circular cylinders.

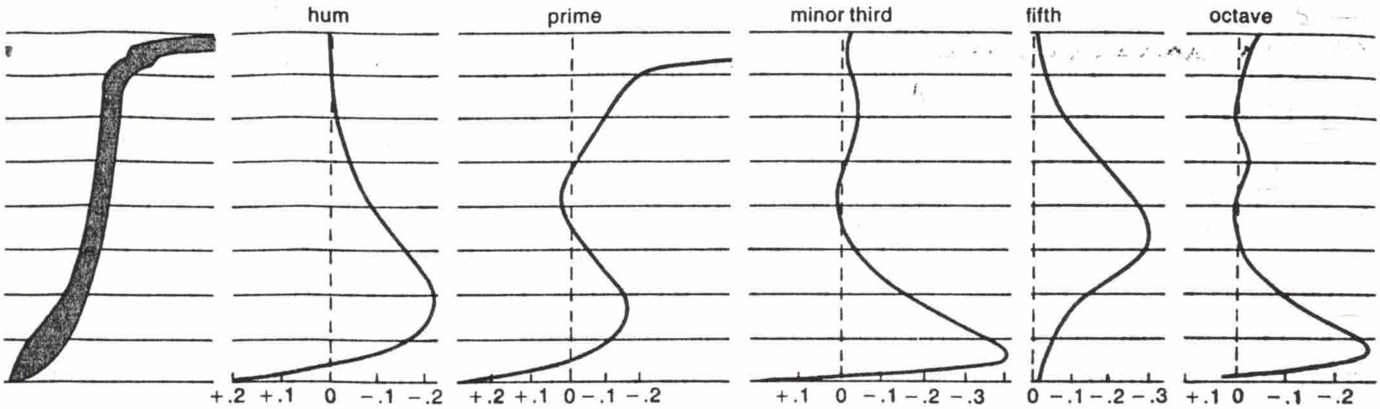

Figure 7. Bell tuning curves show how the first five partials change in frequency when metal is removed from various locations on the inside surface, which appears in cross section at the left. Removing metal from the mouth raises all five partials, whereas a little above the mouth all five are lowered, but by different amounts. Frequency changes are given in tenths of a semitone. (After van Heuven 1949.)

Scaling in handbells is generally less regular. The scale followed by most bell crafters is to make the diameter inversely proportional to the square root of frequency except for the smallest bells, in which the diameter varies inversely with the cube root of frequency (Sathoff and Rossing 1983). The thickness is then adjusted, so that h/d^2 is nearly proportional to frequency.

Bell ringing

There would be little point in making bells and analyzing their sound if it were not for the pleasing musical qualities of their tones. Bells can be rung in many different ways; we conclude this article by briefly describing two rich traditions, change ringing and carillon playing.

Change ringing is an ancient pastime which combines elements of both mathematics and music. It developed in the bell towers of England and later in the colonies, but may be enjoyed indoors with handbells or chimes as well.

The basic strategy of change ringing is to ring a given set of bells in all possible sequences, to move in an orderly fashion from one to another, and to avoid repeating any sequence. There are 24 possible ways to ring four bells in sequence, but with eight bells the number grows to 40,320, and with twelve bells there are 479,001,600 possible sequences, or changes. The number of changes possible is the factorial product ($N!$) of the number of bells.

Change ringing is usually done with four to twelve bells, and each number of bells has a name: four bells are a minimus, five bells doubles, six bells a minor, seven bells triples, eight bells a major, nine bells caters, ten bells a royal, eleven bells cinques, and twelve bells a maximus. A change is said to have been rung when all the bells have been struck one time.

The object of change ringing is to strike as many different changes as possible without any repetition. A peal in minor is a set of no fewer than 5,040 changes. Since only 720 different changes (6!) can be rung on six bells, a peal in minor must include seven complete sets of changes. A peal in major is a set of 5,000 or more changes. This is a small fraction of the 40,320 changes possible on eight bells, and so they must all be different. All peals must begin with rounds and finish with rounds, that is, with the bells striking in their natural order from highest to lowest.

A carillon is a collection of at least 23 bells (two octaves) tuned chromatically and played from a console. Large carillons have a range of five or more octaves. The world's largest carillon is the Rockefeller Memorial Carillon in Riverside Church in New York. Only slightly smaller is the Rockefeller carillon at the University of Chicago. Both carillons were constructed by the Gillett and Johnson foundry in England. A small, portable carillon is shown in Figure 8.

Table 1. Names and relative frequencies of important partials of a tuned church bell or carillon bell

Mode	*Group*	*Name of partial*	*Note*	*Ratio to strike note* "Ideal"	*Ratio to strike note* Bell in Fig. 4	*Decay time (s)*
2,0		hum	D_4	0.5	0.50	52
2,1#	II?	fundamental or prime	D_5	1.0	1.00	16
3,1	I	minor third or tierce	F_5	1.2	1.18	16
3,1#	II	fifth or quint	A_5	1.5	1.51	6
4,1	I	octave or nominal	D_6	2.0	2.00	3
4,1#	II	upper third, major third, or deciem	$F\#_6$	2.5	2.52	1.4
2,2	III	fourth or undeciem	G_6	2.67	2.67	3.6
5,1	I	upper fifth, twelfth, or duodeciem	A_6	3.0	3.01	5
6,1	I	upper octave or double octave	D_7	4.0	4.17	4.2
7,1	I	double undeciem	G_7	5.33	5.45	3
8,1	I	sixth	B_7	6.67	6.80	2
9,1	I	triple octave	D_8	8.0	8.23	

Figure 8. Not all carillons are in lofty towers. This carillon is mounted on a truck which transports it from place to place for concerts, mostly in the Netherlands.

The clappers in carillon bells are connected to the console by a system of cables and levers. Each clapper is controlled by a wooden key or pedal on the console. This gives the carillonneur virtually complete control over the musical dynamics, but it requires rather a high degree of agility.

References

Bigelow, A. L. 1961. *The Acoustically Balanced Carillon.* Princeton Univ. School of Engineering.

Chladni, E. F. F. 1787. *Entdeckungen über die Theorie des Klanges.* Leipzig: Breitkopf und Härtel. Excerpts trans. in *Acoustics: Historical and Philosophical Development*, ed. R. B. Lindsay. Stroudsburg, Pa.: Dowden, Hutchinson and Ross, 1973.

Hanson, V. F., J. H. Carlson, K. M. Papouchado, and N. H. Nielsen. 1976. The Liberty Bell: Composition of the famous failure. *Am. Sci.* 64:614–19.

Jones, A. T. 1928. The vibrations of bells. *Phys. Rev.* 31:1092–1102. Repr. in Rossing 1984.

Lehr, A. 1952. A general bell formula. *Acustica* 2:35–38.

———. 1965. Hedendaages Nederlandse klokkengietkunst (Contemporary Dutch bell-founding art). Publ. No. 7, Neth. Acoust. Soc.: 20–40. English trans. in Rossing 1984.

Perrin, R., and T. Charnley. 1973. Group theory and the bell. *J. Sound Vibr.* 31:411–18. Repr. in Rossing 1984.

Perrin, R., T. Charnley, and H. Banu. 1982. Increasing the lifetime of warble-suppressed bells. *J. Sound Vibr.* 80:298–303.

Perrin, R., T. Charnley, and J. de Pont. 1983. Normal modes of the modern English church bell. *J. Sound Vibr.* 90:29–49. Repr. in Rossing 1984.

Perrin, R., R. W. Peterson, T. D. Rossing, and H. J. Sathoff. 1984. Vibrational modes of a handbell. Norfolk, Va.: Acoust. Soc. Am. Abstract in *J. Acoust. Soc. Am.* 75:S10.

Powell, R. L., and K. A. Stetson. 1965. Interferometric vibration analysis by wavefront reconstruction. *J. Opt. Soc. Am.* 55:1593–98.

Price, P. 1973. Handbells from earliest times. *Overtones* 19(3):12–27. Excerpts repr. in Rossing 1984.

Rayleigh (J. W. Strutt, Lord). 1890. On bells. *Phil. Mag.* 29:1–17. Repr. in Rossing 1984.

———. 1894. *The Theory of Sound*, vol. 1. London: Macmillan. Repr. by Dover 1945.

Ritsma, R. J. 1967. Frequencies dominant in the perception of the pitch of complex sounds. *J. Acoust. Soc. Am.* 42:191–98.

Rossing, T. D. 1981. Acoustics of tuned handbells. *Overtones* 27(1):4–10, 27.

———. 1982. Chladni's law for vibrating plates. *Am. J. Phys.* 50:271–74.

———. 1983. *Science of Sound.* Addison-Wesley.

———. 1984. *Acoustics of Bells.* Van Nostrand Reinhold.

Rossing, T. D., and H. J. Sathoff. 1980. Modes of vibration and sound radiation from tuned handbells. *J. Acoust. Soc. Am.* 68:1600–07.

Sathoff, H. J., and T. D. Rossing. 1983. Scaling of handbells. *J. Acoust. Soc. Am.* 73:2225–26.

Schad, C. R., and H. Warlimont. 1973. Akustiche Untersuchungen zum Einflusz des Werkstoffs auf der Klang von Glocken (Acoustical investigations of the influence of the material on the sound of bells). *Acustica* 29:1–14. English trans. in Rossing 1984.

Slaymaker, F. H. 1970. Chords from tones having stretched partials. *J. Acoust. Soc. Am.* 47:1569–71.

Tyzzer, F. G. 1930. Characteristics of bell vibrations. *J. Franklin Inst.* 210:55–66. Repr. in Rossing 1984.

van Heuven, E. W. 1949. *Acoustical Measurements on Church Bells and Carillons.* 's Gravenhage: De Gebroeders van Cleef. Excerpts repr. in Rossing 1984.