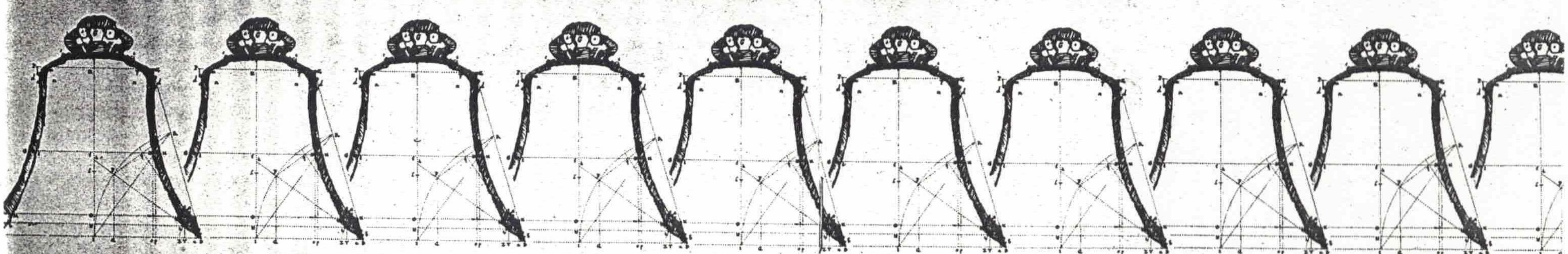
THE DESIGNING OF SWINGING BELLS AND CARILLON BELLS IN THE PAST AND PRESENT

Dr. André Lehr



Athanasius Kircher Foundation
Asten - The Netherlands - 1987

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PREFACE

For several years I have had the intention of writing a short history of designing bell profiles. But it didn't became reality until recently when, with the help of the finite element method, a completely new area was opened in computing bells.

My sincere acknowledgement goes to Mr. Alan Blair at Jülich (Germany), to Dr. Konrad Bund, Archivoberrat of the city of Frankfurt am Main (Germany), and to Mr. Rob Hacquebord of the Royal Eijsbouts Bellfoundry in Asten. After reading the manuscript they gave me many valuable remarks.

But most of all I wish to express my warm thanks to Dr. Thomas D. Rossing, professor of physics at Northern Illinois University, DeKalb (Illinois, U.S.A.), who read the manuscript very carefully on a long train journey through Europe. I am very gratefull for his expert advices!

Last but not least I wish to thank the Athanasius Kircher Foundation which made it possible to publish this treatise.

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1. INTRODUCTION

1.1 The aim of this study

Campanology is the science of the campana or bell. Central to this discipline is the investigation of the relationship between the profile of the bell, that is its half vertical cross-section, and the partials of the bell, that is its hum note and overtones. This is a particularly important feature of campanology, because it is the very character of the partials that determines the quality of the bell sound. This is not just a matter of their frequencies or tones, but also their loudness and reverberation times. It is especially these three elements that are fundamental to determining whether the bell will sound pleasant.

In this study, which has a historical perspective, we will discuss the frequencies of the partials first; not because their loudness and reverberation times are unimportant, but because the primary condition for achieving a particular sound depends on the bell with the lowest partials ringing the desired chord. Then, we will briefly discuss the loudness and reverberation times, in particular of those overtones which can influence the timbre of the bell. These must possess the qualities of ringing sufficiently long and loud for the listener, so the bell timbre as a determinant of quality is not only related to the frequencies, but also to the individual loudness and reverberation times of the partials.

This will be illustrated particularly by the long quest for, and final realization of the major-third bell, in which the traditional minor third has been replaced by a major third. Here should be mentioned already that this shift of only a semitone seems easy to achieve for just one partial, but in reality it is infinitely more complicated than the slight difference suggests. For as different as the tonal structures of the traditional bells may be, and as difficult they are to compare sometimes, it is certain that all these bells have at least one common characteristic: they all show the minor third more or less accurately, in spite of the most bizarre profiles.

1.2 Science and craft

Searching for the oldest historical traces of the problem of the relation between profile and mode frequencies, we soon realize that the science of campanology was not much pursued until the end of the 18th century, or even later. Moreover, what evidence we have indicates that this science was hardly ever based upon what bell founders could reveal about their craft, if they were willing to reveal anything at all, and if they did not lose themselves in quasi-learned treatises.

No, these bell founders were not - and are often still not today - very communicative about their profession. Therefore, the few external scientists that through the ages have practised campanology could not join a campanological tradition, and again had to discover for themselves its fundamental principles. It was a very slow and seldom successful process, because the sporadic investigations did not yield reliable publications until far into the 19th century. Continuity was far from the case.

All this was reinforced by the fact that purely ringing bells were only founded in The Netherlands and Belgium during the 17th and 18th centuries, and after that the necessary knowledge of the principles of campanology was lost. They were not rediscovered until the end of the 19th century in England and were from then on, once again common knowledge in the traditional carillon nations. In spite of this, the number of bellfounders who completely understand the craft of founding and tuning pure ringing bells can still be counted on the fingers of one hand, even in the 20th century!

In the past, the investigator who had no information from bellfounders and certainly not from founders who made purely ringing bells, and who also lived in a country where such bells could not be heard, had a dual problem, even if he did not realize it himself. When, for example, famous acousticians such as Ernst Chladni (1756-1827) or Hermann Helmholtz (1821-1894) wrote about bells, they did so only from their studies and laboratories in Germany - without ever having entered a foundry and without ever having heard a pure-ringing bell. So the starting point for their well-intentioned research was fragile, and thus, inevitably, a source of many misinterpretations!

Yet Chladni and Helmholtz had at least one advantage: they listened to bells in Germany, however poorly tuned their sound mostly was. For them, bells were not only the subject of theoretical contemplations, but also instruments producing sounds, sounds that should be heard if the bell is to be understood. But this was certainly not a common attitude! It was not until the end of the 19th century that Lord Rayleigh (1842-1919) in England, and Father Johannes Blessing in Germany gave new impetus to the scientific study of bells and carillons. In The Netherlands this impetus came from a thesis in 1909 by the physicist Abraham Vas Nunes (1879-1940).[1]

At first, willing but too often bogus experts took the floor. As one of many examples, a certain parish priest Roujoux wrote quite open-mindedly in his rather well-known treatise (1765) on bells that every good bell should possess the sequence of thirds C_1-E_1-G_1-B_1-D_2-F_2-A_2-C_3.[2] How wrong he was, we will show later; but for now we will simply conclude that this priest had either absolutely no affinity for music, or had never listened to the bells in his own church. In the latter case, he could have heard one, or at most two, thirds. So, the realm of the study was very far removed from the village green, where he would never have made such a foolish remark. Indeed, the campanological findings made in the Netherlands in the 17th century had little bearing on that French village!

1. Vas Nunes 1909.
2. Roujoux 1765, p. 4-6.

There was also the philosopher Georg Wilhelm Friedrich Hegel (1770-1831), who in his dissertations on the aesthetics of music assumed that a bell had only a few partials.[3] In this way, he tried to account for the typical sound of a bell. Apparently, he did not really listen to bells either.

3. Hegel 1835, p. 70.

2. THE OBSERVATION OF OVERTONES IN THE PAST

2.1 Measuring partials

Although in this study we will deal primarily with the relationship between profile and mode frequencies, it seems sensible to briefly discuss the methods of measuring the freqencies of partials, and related to that, how the positions of the nodes and antinodes are determined.

It was not until the second half of the 19th century that adjustable and calibrated tuning forks were introduced into bellfoundries (Figure 2.1.1). These tuning forks had adjustable weights along their prongs, producing varying tones according to their position.[4] The range of such a tuning fork is usually between a fourth and a fifth. The procedure with bells is rather simple. The fork is adjusted aurally as accurately as possible to the required partial. Then the base of the fork is placed against the bell, preferably at an antinode, in order to hear if - after the fork is struck - the bell will resonate to the set tone of that partial. The more spontaneous and free of beats the resonance is, the more precisely the fork has been set. An even more accurate method is to strike the fork and bell simultaneously - without contact between fork and bell - and then to count the number of beats. If there are none, they are in unison, and the frequency can be read from the prongs of the tuning fork. These are calibrated to 1/16 of a semitone, the smallest unit thus being slightly more than 6 cents.

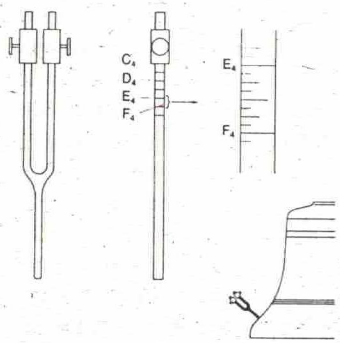

Figure 2.1.1. An adjustable tuning fork (after Lehr 1976, p. 26).

When the adjustable tuning fork was introduced into the foundry, it already had a relatively long history as a fixed source of tones. Invented by Handel's trumpeter John Shore (d.1752), it was developed in 1834 by the German J. Heinrich Scheibler (1777-1837) into the tonometer, a series of fixed tuning-forks differing from each other by only a few hertz, that is to say 65 forks with increasing frequencies in steps of 4 Hz, thus covering the whole range between a C_4 of 256 and a C_5 of 512 Hz.[5] Then the adjustable fork appeared, leading to a dramatic reduction in the number of forks, while covering a wider range of frequencies more easily. Especially in a belfry this was a far from negligible advantage!

This tonometer was also used for a long time in foundries. Although German founders, in particular, used the adjustable forks, by the end of the 19th century English founders preferred a large number of fixed forks to measure each partial by counting the number of beats between the nearest lower and higher tuning fork. It is partly because of this excellent method that the early 20th century carillons from England have become world famous.

Obviously, tuning forks were the best tools for the bellfounders who measured the partials under far from ideal conditions in the workshop, or in the belfry. For research in a laboratory, less manageable techniques were used, in particular because there was not always a need for accurate measurement of the frequency.

For this reason, several techniques were developed in the past to separate the partials in a complex sound without using tuning forks. Apart from the usual analytical listening, with its full emphasis on the trained ear, it was Chladni who around 1800 suggested, holding the rim of the bell at one or more meridian nodes of the desired partial and drawing a violin bow across the rim.[6] Without being disturbed by other partials, one can listen to and measure the overtone produced.

Some fifty years later, spherical resonators, which were easier to handle, came into use. Their inventor Helmholtz said, that when using the right resonator for the required partial, even the most unmusical person could recognize the tone between the other weaker partials.[7] Of course strictly exact measurements were not possible, as the resonance curve of the resonator was too broad. But in many cases less accuracy was of no concern for the investigator. Lord Rayleigh, for instance, used his harmonium to define the tone by the number of beats between a tone on the instrument and the partial of the bell.[8]

Yet the resonance method was always the preferred one. It was known for example, that every individual partial could be produced from a bell if the right tone of the partial was given by human voice, by a flute or any other instrument.[9] Already in the 17th century, the esquire Jacob van Eyck (c.1590-1657), carillonneur in Utrecht (The Netherlands), knew of this, as will be discussed later in more detail. Here we will only mention that, thanks to Van Eyck, this very principle had led to the invention of an excellent measuring instrument in 17th century Holland.[10]

This instrument was composed of a series of metallophone bars that had to be brought into resonance by the partials of the bell. At maximum resonance, the tone of the metallophone and the overtone of the bell were equal. It must be noted that only semitones were attainable, and thus it could only be checked if the relevant overtones fitted into the desired scale of semitones. This is a

4. Llewellins 1879, p. 14-15; Blessing 1894, vol. 19, p. 97-99; Walter 1913, p. 553-554; Griesbacher 1927, p. 2-10.
5. Helmholtz 1865, English translation 1954, p. 443-446; Rayleigh 1877, I, par. 60-61.

6. Chladni 1802, p. 193.
7. Helmholtz 1865, p. 74-75.
8. Rayleigh 1877, part 1, par. 235; Rayleigh 1890.
9. Walter 1913, p. 553.
10. Lehr 1959, p. 46-47; Lehr 1965.

technique that was common in bellfoundries until the end of the 18th century.

Of course, it was not always that easy to determine maximal resonance, as the bell will usually drown the weakly resonating bar. This problem was solved by sprinkling some sand on the bar, then when the desired resonance occured, the sand started to move and finally gathered at the nodes of the bar.

It is amusing to think that Leonardo da Vinci (1452-1519) already saw the necessity of this aid too.[11] He found that one cannot always hear if an instrument brings another instrument into resonance, because of masking. That is why he attached a little straw on the string that had to resonate with another string. If the straw started to vibrate, he knew that resonance was occuring! But let us return to bells.

It is interesting that long before Chladni published his sound patterns in 1787, Dutch founders used the same phenomenon of sprinkling sand to discover whether a bar resonated with a mode frequency of a bell. The same method could be used to determine nodes and antinodes on a bell, as the German investigator Friedrich Melde (1832-1901) did in 1860, still a fascinating experiment.[12] Chladni developed another method as well. He partially filled the bell with water and let the bell ring in only one partial as described above. Thus the nodes and antinodes of the circumference could be seen in the water.

A good forerunner of this method was provided in the beginning of the 18th century by John Theophilus Desaguliere (1683-1744), albeit only for the hum note.[13] He mounted a steel peg very close to the surface of the bell. After striking, the bell rattled against the peg. He concluded that the bell evidently changes its shape during the vibrations and that the round horizontal cross section is constantly changing into oblong circles, as he wrote so expressively.

Later, the nodes and antinodes were determined in a simpler way. Striking the bell surface in several places and checking whether the partial under survey can still be heard is , of course, an excellent method, but fatiguing for the ear. It will be obvious that during determination of the number of meridian nodes (2s) around the circumference, striking should always occur at the same position. For the nodal circles that have fixed positions on the bell surface, this is of course unnecessary. Interesting enough, these nodal lines seemed at first to have been overlooked completely. Helmholtz supposed that they existed, but assumed they had not been investigated until then. Melde shared this opinion, while Otte, not being a physicist, even declared that a bell has no nodal circles at all.[14] In that respect he was in good company, for the aforementioned Vas Nunes had indeed found that there are partials with a parallel, but he doubted, from evidence of aural tests, if overtones with more than one parallel really existed.[15]

Another method of establishing the existence of meridian nodes, although hardly applicable to real bells, was to rotate the bell around its axis, counting how many times the partial under investigation varies in amplitude during one revolution. This number is equal to

$$2s \cdot \frac{s^2-1}{s^2+1}$$

which permits the calculation of the number of nodal lines s.[16] It must be noted that the value s is half the number of nodal lines around the circumference. In other words, the bell is regarded as a bent plate, the nodal lines going from rim to rim via the centre - that is, the point of suspension of the bell.

If the bell is slightly asymmetrical one can, at least for the determination of the number of meridian nodes, successfully use the resulting beats, as Lord Rayleigh did.[17] The procedure is as follows. For an asymmetric bell, each mode frequency will split up into two nearby tones. If there is any extra stiffness in the circumference, the higher one will have one of its antinodes at the position where the stiffness is maximal; the positions of the the lower ones, in contrast, are shifted with an angle of 90/s degrees to those of the higher one. In other words, the higher component has its antinodes where the lower one has its nodes, and vice versa.

If a partial, such as the hum note, has four nodal lines in the circumference (s=2) - and in this case four for each component - then around the circumference eight sites without beats can be found; in each case one component has an antinode and the other, therefore, a node. Apparently, the number of meridian nodes is half that of the number of positions without beats. But once again it must be noted that this approach can only be used if the partial is split into two tones that are close together.

2.2 Campanology in The Netherlands in the 17th century

It is not obvious to everyone that the individual partials can be noticed by themselves. Many listeners are especially fascinated by the total sound. Others have a natural tendency towards analysis and have no great difficulties in hearing the individual overtones one by one. This process is, by the way, easier if the partial listened to forms a less consonant interval with the main tone. Two tones with an octave interval (frequency ratio 2/1) are much more difficult to discriminate than two tones with an interval of a full tone (9/8), for example. The bell also has some of those wayward partials. The major one is the loudly sounding minor-third partial E^b (6/5) which strongly influences the character of the bell and by tradition is considered to be the characteristic partial of the bell. There are musicians and listeners that prefer for this a major-third E (5/4) to eliminate what to their ears is the musically isolated position of the bell. This will be discussed later on.

It is doubtful whether many analytical observations have been made in the

11. Truesdell 1960, p. 19.
12. Melde 1860.
13. Desaguliers 1746, p. 16-18.
14. Chladni 1802, p. 192-197; Otte 1858, p. 57; Helmholtz 1865, p. 125; Winkelmann 1909, part 1, p. 407.
15. Vas Nunes 1909, p. 77.
16. Rayleigh 1877, part 1, par. 233; Winkelmann 1909, p. 406.
17. Rayleigh 1877, part 1, par 235; Llewellins 1879, p. 13, also knew of this phenomenon.

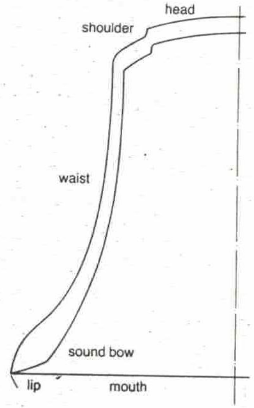

Figure 2.2.1. The parts of a bell.

past; at least until the end of the 17th century, reports of them are scarce. Vincent de Beauvais (d.1250) wrote that in a bell three individual tones can be produced: a low one by striking the bell in the middle, a high one by striking at the upper waist, and an intermediate one by striking at the lower waist (Figure 2.2.1).[18] In anticipation of later considerations we will now call these partials by their names, names that for a purely tuned bell are similar to the intervals between them. If we define the lowest tone in the tonal spectrum as C_4, it will almost certainly be the following partials: hum note C_4, minor-third E^b_5, and nominal C_6. In a well-designed and well-tuned bell the mutual frequency ratios are 5:12:20. By the way, this picture is incomplete. In the lower part of the tonal spectrum more partials exist than are mentioned here (Section 3.3).

So the observation of Vincent de Beauvais was a rather isolated one in the course of history. This is not surprising, since scientific interest in the phenomenon of overtones did not develop until the beginning of the 17th century.[19] Campanology did know fruitful times during the life of the musician and esquire Jacob van Eyck of Utrecht, who, on one hand was professionally concerned with the daily practice of music, and on the other hand had important connections with prominent scientists of his time, Christiaan Huygens (1629-1695) being a distant relative of him. Some other scientists interested in the overtones of the bell were Isaac Beeckman (1588-1637), Marin Mersenne (1588-1648) and René Descartes (1596-1650). Although they had no solution for the problem of how the overtones should be positioned in a good-sounding bell, they did succeed in creating the scientific climate in which the solution could be found.

The above-mentioned carillonneur Van Eyck was finally successful, thanks to his personal dedication and his contacts in both the world of the carillon and of science. He was able to interest the very capable bellfounders François (c.1609-1667) and Pieter (1619-1680) Hemony in Zutphen (The Netherlands) in his investigations, resulting in a cooperation that ultimately led to the founding of the first really well-tuned carillon in history![20]

It was in this unique situation that Van Eyck, considering the technical possibilities of the bell, was the first to formulate which intervals the lowest partials of a bell should form together.[21] Moreover, headopted as a particular reference point a tone which is dominant during the striking, to which all other tones could be referred by trained hearing. This tone, which he called the strike, was known by founders as the cornet. Below the strongest sounding partial, two octaves should be found: a minor-third and a fifth. If we call the strike or cornet C_5 then the desired series is: C_4-C_5-E^b_5-G_5-C_6. Nowadays, they have the names: hum note, fundamental, minor-third, fifth, and nominal, while the strike or cornet is called the strike note.

It is remarkable that these partials keep their names, even when the intervals in less well sounding bells are aberrant. If, for example, the fundamental is an entire tone too low - thus is a B^b_4 - it is still considered as the fundamental partial, instead of a minor-seventh partial or even a doubly-diminished fundamental. Obviously, for laypersons this is confusing terminology.

One should note that it was not the hum note, but the partial two octaves above that was considered as the most important. But there is more to say about the ideal bell, as defined by Van Eyck and finally realized by the Hemony brothers in 1644. This bell had a minor third amongst its lower overtones. Beeckman wrote about this to Mersenne, to whom it was far from obvious. He emphasized that a bell does not sound do-mi-sol, but re-fa-la. Van Eyck remarked in his analysis, sensational for that time, that nobody knew why only these partials can be realized in a bell. Only God knew. Now we have somewhat more knowledge and know that the minor-third partial is the result of the global profile curves that were already set in the Middle Ages, as will be discussed later in detail.

It is hardly necessary to say that the campanological findings of Van Eyck were of utmost significance. Not only did he accurately perceive the overtones of the bell, but he also sketched a musically ideal bell. At the time of his research this was certainly not superfluous, as Van Eyck realized. He told Beeckman, for example, that the middle octave (the fundamental C_5) was usually false. Indeed, many of the old swinging bells in Western Europe often display the usual pattern of having a fundamental that is sometimes too high but usually too low, assuming for convenience - albeit questionable - that the other tones are right (See also Table 3.4.1). Numerous variants on this false scheme existed, and it was to Van Eyck's credit that he was able to formulate precisely which partials occur in a bell, and how they can be arranged, within the possibilities of the traditional bell profile, to be musically acceptable. This subject will also be discussed later on.

How did Van Eyck arrive at his discovery of the overtones? Christiaan Huygens described this in detail. Using a wine glass, Van Eyck demonstrated how, by whistling the right tone, the partials of the glass can be made to sound. A glass was also found to have several partials. The same he demonstrated with the partials in the bells of Utrecht Cathedral. Huygens was deeply impressed and called Van Eyck a man with a striking hearing and judgement.[22]

18. Otte 1858, p. 54, n.1; Theobald 1933, p. 405. "Campana in tribus locis, si pulsetur tres habere sonor invenitr, in fundo mediorem, in extremitate subtiliorem, in medio graviorem." Vincent de Beauvais had some other interesting remarks about bells, too (Otte 1858, p. 56 and 60).
19. Farrer 1956; Palisca 1961; Cohen 1984.
20. See for these founders e.g. Lehr 1959.
21. Worp 1913, vol. 2, p. 487-488; De Waard 1942, p. 329-330; De Waard 1945, p. 310-311.
22. Jonckbloet and Land 1882, Correspondance, p. 6-7; Worp 1913, vol. 2, p. 487-488. Huygens thought that both a glass and a bell had three partials (1937, vol. 19, p. 361-365).

In the contacts Mersenne had - through others - with Van Eyck,[23] the former claims to have understood that a bell should have three well-tuned octave partials. The hum note can be heard at the sound bow, the fundamental at the lower waist (which seems to be wrong, as this is immediately above the nodal circle of the fundamental), and the nominal at the upper waist. False bells result from wrong arcs, thus from wrong profiles. He knew, by the way, where these octaves should be tuned - that is, where inside of the bell the wall can be made thinner to correct small tonal aberrations. Those partials can be produced in the same way as in a wine glass, by whistling the appropriate tone.[24]

Later, François Hemony was even more specific in a letter of February 26, 1653 to no less a person than Athanasius Kircher (1602-1680): "A well-designed bell should have such proportions that three octaves, two fifths, a minor and major third (and not "or" as is often translated) can be heard. One of these tones can be called the major tone, the highest tone of these octaves, as this one sounds much more clearly and dominates the others tones, which are subordinate."[25] Formulated from our present knowledge these would be the notes: C_4-C_5-$E^{\flat}_5$-G_5-C_6-E_6-G_6-C_7, and by name: hum note - fundamental - minor third - fifth - nominal - major tenth - twelfth - double octave (sixteenth). This corresponds to the frequency ratios 5:10:12:15:20:25:30:40, which can not be simplified without introducing fractions.

It is doubtful whether the discoveries of Van Eyck, put into effect by the Hemony brothers, became widely known in the scientific world. Even in their local surroundings, often only three tones were accepted in a bell, analogous to the partials heard in a wine glass. This misunderstanding, prevalent until far into the 18th century,[26] was also caused by the three tones found by Vincent de Beauvais in the bell. This is not in the least surprising because many authors, according to their scientific traditions, preferred to quote existing publications rather than listening to a real bell themselves.

It is no wonder, therefore, that outside The Netherlands, and long after the 17th century, the fundamentals of campanology had to be discovered again and again, albeit with little success. Later, we will come across more evidence to support this. Here we are satisfied to say that, only at the end of the 19th century, such investigators as Lord Rayleigh in England and Johannes Blessing in Germany start scientific campanology, at the same time that Canon Arthur Simpson (1828-1900) in England rediscovered the art of tuning.

23. Mersenne 1636, marginal notes.
24. Mersenne 1636, margnal notes.
25. Schott 1674, p. 356-360.
26. Otte 1858, p. 54-55.

3. THE OVERTONE STRUCTURE OF THE BELL

3.1 Extensional and inextensional vibrations in the bell

Before discussing the musical aspects of the mode frequencies of bells, we have to face the question of the dynamic characteristics of the vibrational forms for the musically important partials. Since the end of the 19th century, when Lord Rayleigh expressed a clear opinion about the matter, there seemed to have been few doubts. Yet even in his lifetime, there was criticism of his ideas. So what is the case?

If one assumes that a bell, as a vibrating medium, is a conservative system in which no external forces act and in which no damping occurs, then at any given moment the sum of the kinetic energy T, which comprises the square of the frequency f, plus the potential energy V will remain the same. From this principle Rayleigh computed the mode frequencies either by assuming that in the medium the maximal kinetic energy (at the passage through equilibrium point) is equal to the maximal potential energy (at the extreme positions), or by putting both expressions in the well-known equation of Lagrange.

Apart from the mathematical complications, it is the potential energy V that poses some physical problems. For an arbitrary part of a vibrating shell, V has to be composed of two parts: one term V_1 resulting from the periodic bending of the element (changes in shape perpendicular to the surface), another term V_2 resulting from longitudinal stretching and shrinking of the element. For pure bending vibrations, with $V_2=0$, the middle plane of the vibrating shell will neither stretch nor shrink. But if $V_1=0$, as in the case of pure extensional vibrations, there is no bending, and we speak - for example in a bar - of longitudinal vibrations.

Mechanical considerations show that the kinetic energy T is proportional to the thickness d of the shell. For the potential energy V the bending part V_1 is proportional to the third power of the thickness, whereas the stretch part V_2 is proportional to the thickness. This means that:

$$T_{max}(f^2,d) = V_{max}(V_1(d^3) + V_2(d))$$

In other words, for pure inextensional vibrations the mode frequencies are proportional to the thickness, whereas the frequencies of the pure extensional vibrations are independent of the thickness.

Rayleigh now assumed that the lowest mode frequencies of a shell, such as a bell, were pure inextensional vibrations,[27] but Love did not agree.[28] In his opinion it was incorrect to ignore the term V_2 in favour of the term V_1, both being important. This also means that according to Love, the frequencies of the partials can indeed be directly proportional to the thickness, but this relation-

27. Rayleigh 1877, vol. 1, par. 235b. See also Kalnins & Dym 1976, p. 142-147.
28. Love 1888.

ship will differ for each partial or group of partials. History proved him to the right, because both later theoretical considerations[29] and the measurements of bells with varying thickness[30] confirm Love's idea.

Vibrations with s=0 and s=1 can be either extensional or inextensional vibrations.[31] They are usually high in the sound spectrum. Only in thick bells, that is to say bells for which one-fourth the product of the diameter (m) and the frequency of the nominal (Hz) - the so-called f·D-value - exceeds 250 m/s, they will be on such a low level in the spectrum that their existence cannot be completely ignored.[32] Increasing the thickness raises the inextensional, but not the extensional vibrations, so that the position of the latter will be relatively lower in the sound spectrum.

3.2 Harmonic and non harmonic overtones

Frequently, in the past, the question was put why the frequencies of the partials in a bell do not have harmonic ratios (i.e., according to a series of integers 1:2:3:4:5:6, etc). Two reasons exist for this musically disturbing fact.

A completely elastic string possesses harmonic overtones. If a completely elastic vibrating medium has more degrees or freedom than one (in other words the waves can propagate in more than one direction), the harmonic structure will be lost. A membrane in the form of a drum head will usually have only a few harmonic ratios because of its two-dimensional character. Only an intervention of its maker, for instance by choosing a harmonically favourable ratio between length and width - e.g. 3/2 (fifth) - can change this.

Let us return to the one-dimensional string. Two extreme possibilities can be discerned: a completely elastic string in which the restoring force that brings the string back into its original position is the tension, or a bar where its own stiffness brings the bar back into the equilibrium position. There are of course many intermediate types. The not-completely flexible string is an example of a one-dimensional medium.

The equation expressing the movement of a completely flexible string is a second-order differential equation, but for a one-dimensional bar it is of fourth order. Consequently, the solution of that equation is not a type of vibration where the speed of the waves is independent of the frequency. In other words, such a bar shows dispersion.

The conclusion must be that harmonic overtones can occur only if the vibrations propagate in just one direction, and if the vibrating medium is completely

29. Ross 1968 in Kalnins & Dym 1976, p. 155-162.
30. Lehr 1986.
31. See Figure 4 in Rossing and Perrin 1987.
32. Originally the f·D-value was defined by the product of the diameter and the frequency of the hum note (Van Heuven 1949, p. 43). This holds only for well-tuned octave bells. In other type of bells the hum note can deviate so strongly from an octave that it seems better to define the f·D-value with help of the most important and reference partial of a bell, the nominal. In well-tuned bells this overtone is two octaves higher than the hum note, thus four times higher in frequency.

elastic, with no stiffness of its own. The bell meets neither of these conditions, and therefore belongs, in both respects, to the family of musical instruments with inharmonic overtones, at least as long as the maker makes no modifications that change its characteristics.

3.3 Classification into groups

Although at first sight one tends to think that each change in profile changes the frequency of each partial in a strictly individual way, this idea proves to be incorrect. On the other hand, it is possible to demonstrate that groups of partials react in an almost identical way to a model change. The reason is, without a doubt, that within a group of partials the same type of vertical mode of vibration is valid.[33]

Before we discuss these modes in more detail, we point out an interesting and important consequence. It turns out that for large profile changes, the intervals between the partials in the same group - albeit with the unexplained exception of the lowest of each group (s=2) - only change a little, whereas the shift of the relative positions between the groups for the partials with four meridian nodes (s=2) is much larger. This unusual phenomenon, which prevented the discovery of a major-third bell, will be brought up again later.

For each inextensional vibration mode we understand a function which gives in a relative way the largest radial amplitude of the partial for each position at a meridian on the surface of the bell. Although an inextensional vibration, in addition to the radial motion, also has motions parallel to the surface of the bell - at least if the conditions of the completely or partly non-stretching and non-bending middle plane are met - the tangential motions are, from a musical point of view, unimportant, as they do not contribute to the sound of the bell.

The possibility of using the mode of vibration to determine in which group a partial belongs, is restricted to the radial motion along a meridian, which in principle can be arbitrary, as long as it is not a nodal meridian. The choice for meridian nodes is therefore also plausible, as the motion of a partial along a parallel can be described using a sine or cosine function. However, it will be clear that such a function is not suitable for identifying groups of partials with nearly identical profile characteristics. That is why the vertical one is always meant when speaking of the vibration pattern of a partial.

Numbering the groups according to their vertical vibration pattern with the Roman numbers I, II etc., a survey can be made of the West-European bell, as shown in Table 3.3.1 with reference to Figure 3.3.1. In this case it is a bell with

33. Lehr 1965; Lehr 1976, p. 31-42; Lehr 1986. It can be noted that the same type of groups exists, for instance, in handbells (Rossing and Sathoff 1980; Rossing, Perrin, Sathoff and Peterson 1984; Rossing and Perrin 1987, p. 45-55). On the other hand, there are some differences, especially in the excistence of nodal circles for the lowest partials of the first group. Experiments at the Royal Eijsbouts bellfoundry have learned that the presence of these nodal lines depends on how complex the profile is.

musical name	note	s	group	physical name
hum note	F_3-43	2	I	I-2
fundamental	F_4-41	2	II	II-2
minor third	A^b_4-27	3	I	I-3
fifth	C_5-36	3	II	II-3
nominal	F_5-41	4	I	I-4
major tenth	A_5-7	4	II	II-4
1st eleventh	B^b_5-81	2	III	III-2
2nd eleventh	B^b_5-46	3	III	III-3
twelfth	C_6-34	5	I	I-5
thirteenth	D_6-64	4	III	III-4
fourteenth	E_6-39	5	II	II-5
double octave	F_6+32	6	I	I-6
	F_6+34	2	IV	IV-2
	$F^\#_6-35$	3	IV	IV-3
	G_6-45	5	III	III-5
	A^b_6-40	4	IV	IV-4
	A_6+7	6	II	II-6
fourth from	B^b_6+1	7	I	I-7
double octave)	B_6-22	5	IV	IV-5
	B_6+39	6	III	III-6
	B_6+45	2	V	V-2
	C_7+30	3	V	V-3
	$C^\#_7+21$	4	V	V-4
	$C^\#_7+42$	7	II	II-7
sixth from	D_7-14	8	I	I-8
double oct.)	D_7+32	6	IV	IV-6

Table 3.3.1. Partial groups in the bell sound.

a diameter of 116 cm (45 11/16"). The deviations are given in cents. The table of the inextensional modes of vibration is limited to IV-6, but of course there are many higher overtones, not only of groups I-V but of group VI and higher too.[34]

It is clear that one can choose another numbering for the partials too, for example, giving two numbers: nodal meridians and nodal circles.[35] But this is less satisfactory, because only with a classification into groups the musical significance of groups of partials can be easily mphasized. In this way, the partials of group one, for example, which have no nodal circle in the sound bow, are recognizable.

Note that the two lowest partials, the hum note and the fundamental, cannot be classified into a particular group, at least if we adhere strictly to their vibration patterns. However, other criteria exist. A negative one is the consideration that if the two partials cannot be classified in a group, both the first and second group lack the vibration pattern with s=2. This is why for the sake of unity

34. Lehr 1986.
35. Tyzzer 1930; Rossing 1984.

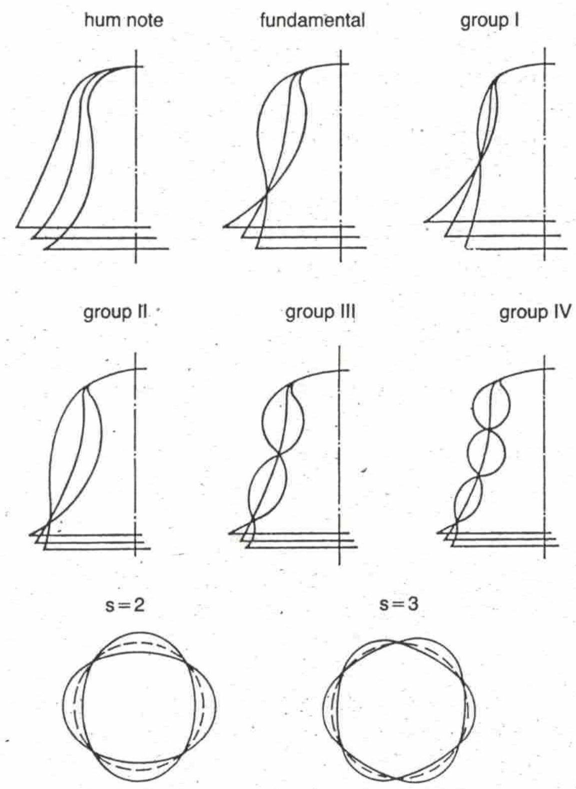

Figure 3.3.1. Vertical and horizontal vibration patterns of bell partials (after Lehr 1976, p. 30–32).

in nomenclature the hum note is indicated by I-2 and the fundamental by II-2. On the other hand one could number the hum note and the fundamental as an individual group zero.[36] But the fact remains that from a musical point of view both partials do not have the same significance in the sound spectrum of a bell (See also Section 7.3).

Using this classification some words can be said about the sound spectrum of the bell. Tones that have a nodal circle at the sound bow - that is to say all partials of the second and higher groups, except for the fundamental II-2 - will sound weak after being struck. In contrast, all tones of the first group will dominate: C_4-E^b_5-C_6-G_6-C_7+ etc. This helps in understanding the curious phenomenon of the strike note. The bell is heard by the common periodicity of the partials C_6-G_6-C_7+, of which the double octave or fifthteenth (C_7+) is least important.[37] This periodicity leads to the strike note C_5, a perceived sound that

36. Rossing and Perrin 1987.
37. Lehr 1986, p. 2010-2011; Eggen 1986.

is not identical to the fundamental II-2, which for pure bells is also at C_5.

It must be noted that the frequency of this periodicity is so strongly dictated by the loudly sounding C_6 that, since the 19th century, a rule of thumb for bellfounders is that the strike note is always an octave interval lower than the partial I-4, i.e. the nominal C_6.[38] Thus, a fixed relationship was presupposed between the strike note and the octave partial. The same fact helped to explain why that strong overtone was accepted, in the 17th century, as the pitch of the bell. Until then, there was no talk of a strike note. In this respect one may remember from paragraph 2.2 how Van Eyck spoke about the strike, and the bellfounders about the cornet. They were not aware of the octave shift when perceiving the bell frequencies.

3.4 Different types of bells in Western Europe

In the preceding paragraph the tonal structure of the West European bell was given with the explicit stipulation that it should be well designed and well tuned. These conditions were certainly not always met. In the case that, because of an ill-designed profile, not only the tonal structure was wrong, but also tuning was impossible, two solutions were available for the founder. Either he kept on trying until he chanced upon a good cast, thus accepting the bells with aberrant tonal structures to be failures, or he raised the aberrant tonal structure to a goal in itself. The latter obviously is a deceptive solution, albeit nolens volens, to solve a vexing question!

The bell in Table 3.3.1 is called an octave bell because of the dominance of octave intervals between I-2 and II-2, II-2 and I-4 and, to a lesser extent, between I-4 and I-6. "And why," the bellfounder and/or expert might have reasoned, when he was not able to realize the octave bell - or didn't even know about its existence - "why can't I name my bells after the interval between the hum note I-2 and the strike note?"[39]

Is such a description sufficient? It seems not, as not only the hum note I-2 and the strike note, but also the nominal I-4, the fundamental II-2, and the minor-third I-3 can be heard loudly. It is conceivable that these vary according to the profile, too. Yet this rather plausible assumption is incorrect, at least for the third I-3. It turns out that it does not matter how the bell profile is formed, at least if we stay within the geometrical limits of the European swinging bell, the interval between the partials I-3 and I-4 is almost always a major sixth. In other words, the interval between the strike note and the third is almost always a minor third.[40]

As the fifth II-3 in a bell is always weak and contributes little to the overall sound, one could classify an arbitrary swinging bell by the positions of the hum note I-2, the fundamental II-2 and the nominal I-4,(i.e., by the strike note).

38. Rayleigh 1877, vol. 1, par. 235a; Rayleigh 1890.
39. Griesbacher 1927, p. 14-39; Weissenbäck und Pfundner 1961, p. 19-26.
40. Lehr 1986 (Musica) from which also other details are taken.

bell type	tonal structure I-2	II-2	I-4	%
octave bell with perfect fundamental	C_4	C_5	C_6	17.4
octave bell with diminished fundamental	C_4	B_4	C_6	14.3
minor-ninth bell with perfect fundamental	B_3	C_5	C_6	7.4
minor-ninth bell with augmented fundamental	B_3	$C_5^\#$	C_6	7.2
octave bell with augmented fundamental	C_4	$C_5^\#$	C_6	6.9
major-seventh bell with perfect fundamental	$C_4^\#$	C_5	C_6	5.8
major-seventh bell with diminished fundamental	$C_4^\#$	B_4	C_6	5.5
minor-ninth with diminished fundamental	B_3	B_4	C_6	4.7
major-seventh bell with double diminished fundamental	$C_4^\#$	B_4^b	C_6	4.1
octave bell with double diminished fundamental	C_4	B_4^b	C_6	3.6
minor-seventh bell with diminished fundamental	D_4	B_4	C_6	2.7
minor-seventh bell with double diminished fundamental	D_4	B_4^b	C_6	2.5
minor-seventh bell with triple diminished fundamental	D_4	A_4	C_6	2.5

Table 3.4.1. Division of 363 West European swinging bells into several types.

And that is why one could classify a bell by the interval between the hum note and the strike note, also noting whether the fundamental is too low or too high. The survey in Table 3.4.1 resulted from a statistical investigation of 363 of West European swinging bells, not intended to be carillon bells, which normally have well tuned octaves between the main partials.

The following can be noted. In these tonal structures of different bell types, the third and fifth are left out for the above-mentioned reasons. Therefore we restrict ourselves to the hum note, the fundamental and the nominal. The latter is chosen as a reference and is noted as a C_6. Therefore, a minor seventh bell will have as hum note D_4, for example. In practice the D_4 is never reached precisely, so we call all bells that have a hum note between D_4-50 and D_4+49 cents a minor-seventh bell. The same rules apply, of course, to other bell types, and for assigning the position of the fundamental. Finally, we remark that we have restricted ourselves to 85% of the available statistical material, thus covering all bell types representing 2% and more of the total.

There are more types of bells, of course, but they occur seldomly as mentioned previously. In particular this is true of the major-sixth bell with a tonal structure E_4^b-B_4^b-E_5^b-(II-3)-C_6, which occured in the statistical material less than 1%, in spite of the fact that at least harmonically their partials are not that unpleasantly grouped! Apparently there are several ways to judge a bell.

One gains the impression, especially from the German literature, that too high a fundamental is not appreciated at all. On the other hand, this seems a bit unrealistic, as the Dutch founder Geert van Wou (c.1450-1527), who was also much admired in Germany, supplied the city of Utrecht (The Netherlands) in 1505 with major-ninth bells having augmented fundamentals.[41] One can hardly suppose that the city was happy with all these aberrant sounds, although whether the bellfounder, musicians and laymen were aware of their aberrant

41. Lehr 1980.

character is another question. It is only certain that seventh bells were less appreciated than sixth bells.[42] But possibly other criteria are valid.

Other opinions existed, however. The German Canon Heinrich Böckeler in 1882 remarked that the medieval bell founders did not strive as much for a pure bell as for a pleasant sounding one. Besides, by allowing several tones not related by octaves, it seems as if more bells are sounding simultaneously. Every day he heard, for example, a bell in which the strike note - using the modern names - was an A_4.[43] But at the same time this bell sounded A_3, G_4 and C_5, which he found beautiful. Apparently this was a bell with a tonal structure A_3-G_4-C_5-(II-3)-A_5, thus an octave bell with doubly diminished fundamental.

It is wrong to suppose that this approach to the sound of the bell is completely outdated. If one heard, for instance in Spain, chiming of stationary bells or bells that are swung upside down, they would not only reveal different habits, but also completely different timbres, which to our ears would sound very dissonant. On the other hand, in Spain or in any other Mediterranean country, the northern bells with their more consonant sounds will be rejected. Similarly, for certain Russian bells the tonal structure C_4-B^b_4-D_5/E^b_5-F_5-C_6 is preferred. Only these bells remind some emigrants and their families in the United States of Russia, and not the musically more balanced Western bells!

The growing awareness of the pure octave swinging bell can therefore only be found in and immediately around The Netherlands. This development began relatively late. Finally it is remarkable that they were neither laymen nor bellfounders who asked for the octave bell, but musicians. They, and nobody else popularised this type of bell. It is not surprising that sometimes the almost impossible was demanded. It was Biehle, for example, who definitely noted that he had never heard an absolutely perfect bell.[44] He would have liked a somewhat prettier arrangement of partials, and since he was asking, he wondered why the natural tones could not be realized in a bell, i.e. tones with frequency ratio's 1:2:3:4 etc.[45] The inevitable question of whether the bellfounder was free to design bells was never asked; even Rayleigh believed that the partials of the bell could be arranged in a harmonic order.[46]

3.5 Why does a bell have a minor-third partial?

The carillon bell, as we know it since the 17th century in The Netherlands and Flanders (Belgium), has between its strongly sounding partials fundamental II-2 and nominal I-4 a clearly audible minor-third partial I-3. Its presence is indeed conspicuous, to the extent that this overtone was accepted as the characteristic tone of a bell. Even innumerous less successful West European bells this partial is dominant as a minor third of the strike note. Thus, whatever its

42. Griesbacher 1927, p. 17-18.
43. Böckeler 1882, p. 119-121.
44. Biehle 1918, p. 30.
45. Biehle 1918; Neumann 1950, p. 59.
46. Rayleigh 1890, p. 12.

imperfections may be, the West European bell is a minor-third bell. But why?

There is no lack of answers. The Austrian Canon Andreas Weissenbäck (d.1960), stated at a meeting of bell experts and founders in the German city of Limburg an der Lahn in 1951, that in the Middle Ages, the minor-third bell was chosen for its beauty of sound.[47] Apparently, he supposed that a free choice had been possible, and it did not came to his mind that for a genuine choice the major-third bell should also have been present. This was certainly not the case. Apparently for him, and for many others, it was absolutely unimportant - or it was completely out of his scope - whether those bellfounders had in fact made several types of bells before selecting the very reliable minor-third bell as the ideal.

Instead of asking these obvious questions, he continued his theorizing by adding to that preference a historical foundation. This was based on two historical facts, namely that bellfounders were monks until far into the Middle Ages, and that the first octave bell was made around 1200 - albeit by accidence and far from perfect. His further speculative reasoning, however, had to support his theory.

Because those founders in the 10th - 12th century were educated monks, they knew everything about music theory, including the Dorian and Phrygian scales that best resemble our minor-third scale. Because of their Romanic traditions they would have preferred to keep to these scales, and so they considered the minor third in a bell to be ideal, though they knew very well that the major third was a more perfect interval. Needless to say, this reasoning is only a specious attempt to argue a free choice that did not exist. Canon Weissenbäck did not understand that the reason might be purely physical.

Weissenbäck was not alone in this line of reasoning. In the same year 1951, it was the far from insignificant Danish expert Rung-Keller, who sought an explanation for the fact that partial I-3 is a minor third and partial II-4 is a major tenth, following a similar path.[48] Isn't that actually a musical anomaly? He supposed that the bellfounders, particularly the famous brothers François and Pieter Hemony, who designed and founded such bells had introduced this discord. In that way, he claimed, these excellent Dutch founders tried to accommodate the fact that music played in major would be intolerable if the bell only sounds in minor. For that reason the Hemony brothers might have introduced a major third in the higher parts of the tonal spectrum, in the form of a major tenth II-4, to support music played in major also from the inner tuning of the bell!

We will not pursue further these suggestions that do not - and cannot - support the facts, as we will show later. Here we are contented to observe that however characteristically the minor partial is perceived, the problem of the third overtone has been posited. And still we have to answer the question why all West European bells have a reasonably or pure minor third. The answer

47. Andreas Weissenbäck at the Deutscher Glockentag at Limburg an der Lahn, 6-9 June 1951, in a lecture titled "Das Wesen der Glocke". This hypothesis was adopted by Ellerhorst 1957, p. 170.
48. Rung-Keller 1951.

seems too simple, as in the first place a bell with the global profile curves of the European bell always turns out to have a more or less pure minor third. Moreover, as both I-3 and I-4 belong to the same group of partials, even relatively large changes in profile will have relatively little effect on the interval between these two overtones, at least within the pattern of the West European bell. Apparently extreme changes in profile, not belonging to the tradition of the bell, are needed to position I-3 at a full tone or major third, not to mention even larger shifts or shifts in the other direction.

Let us now return to the minor third as a problem in melodies in major modes. A suitable and obvious solution seemed to give the third a neutral character! It was Rung-Keller who suggested positioning the bell third just between the minor and major, so this "bastard third", as he called it, would be equally dissonant in both major and minor keys. The Flemish expert Victor van Geyseghem was also an advocate of a neutral third partial. The Royal Eijsbouts bellfoundry even put this into effect, for example, in the carillon it founded in 1955 for the Town Hall of Zeist near Utrecht. Unfortunately, it was not much of a success. For some listeners the carillon music acquired an unpleasant roughness from that neutral third, in all minor- and major-third concords that were played on these bells. There are better ideas to follow.[49]

How about the carillonneurs? Didn't they avoid the discrepancy between the minor-third bell and melodies in major by playing exclusively, or at least preferably, in minor? This is an obvious idea that has been suggested especially in countries far from the centre of the carillon culture in The Netherlands.[50] The answer must be negative. Surely, the carillonneur will take into account the minor character of his bells, but it is too bold a statement to say that he would avoid the major key for this reason. No, the problem of the minor third was not solved by him either.

Clearly, one could think of changing the minor partial to major, hoping for less interference from the major overtone. This idea emerged from quite a different place than the world of the carillon, namely the world of swinging bells. There the problem was also met in a bell giving the opening of the Maria antiphon Salve Regina, which required the bells C_4-E_4-G_4-A_4. Why not a C_4 bell with a major third instead of a minor third?

Applying major thirds in a carillon, too, seems only to move the problem, for in that case one has to object to melodies in minor played on major-third bells. There is more to say, however. In the first place a bell with a major-third overtone will combine much better with other musical instruments having at the 5th harmonic of their fundamental a major third, albeit a weak one! Nevertheless, the advantage for the bell seems evident, in spite of the fact that like the minor third, it will also remain conspicuous. A bell with C_4-C_5-E_5-G_5-C_6-E_6-G_6-C_7, having frequency ratios 2:4:5:6:8:10:12:16 is definitively more consonant than a bell with C_4-C_5-$E\flat_5$-G_5-C_6-E_6-G_6-C_7, having ratios 5:10:12: 15:20:25:30: 40. In that way the always rather dissonant character of a bell

49. Lehr 1971/1981, p. 356.
50. Weissenbäck und Pfunder 1961, p. 87.

Figure 3.5.1. Scores of the two melodies which were used in the experiment (after Houtsma and Tholen, 1987).

could be tempered and made milder. But there is still more.

At the Institute for Perception Research of the Technical University of Eindhoven, research was done using two sets of computer-generated bells with minor and major partials.[51] Using appropriate inquiry techniques listeners were asked which of combination of melody (Figure 3.5.1) and bell type they liked most:

1. melody in minor on minor bells,
2. melody in major on minor bells,
3. melody in minor on major bells,
4. melody in major on major bells.

The results were very surprising and interesting! It was found that (student) carillonneurs preferred minor bells, whatever the key they were played in. In contrast, other musicians selected the combinations of minor melody on minor bells and major melody on major bells. The listener "in the street" had an opinion opposite to the carillonneurs: in all cases they chose major bells, whatever the key they were played in! In Chapter 7 this will be discussed extensively.

Now we remark that it is evident that carillonneurs take into account the minor character of bells, yet without avoiding the minor scale. So, they would rather not use the major third on the hum note in the final chord for obvious reasons. Generally speaking, the carillonneur will take into account these minor-third overtones, avoiding disturbing chords caused by this partial, or in contrast making use of them! In this respect it seems that these investigations in Eindhoven, however valuable they may be, are not actually representative of the art of the carillon. On the other hand, we must realize that in the desired adaption of carillon music to the character of the bells, we report here an ideal situation that in the daily practice of music is seldom realized. But even with an

51. Houtsma en Tholen 1987.

optimal adaption, it is not to be expected that all problems introduced by the minor third will be solved.

3.6 The first steps toward to a major-third bell

The question of whether a bell sounds with a major partial or not, requires more than superficial listening. Suppose we hear this bell: C_4-B_4-$D_5^{\#}$/E_5^{b}-$F_5^{\#}$-C_6, a bell not difficult to realize. Has this bell a major third, as the interval between the fundamental B_4 and the third $D_5^{\#}$ indeed is of that distance? Or should we speak of a minor bell, as the E_5^{b} partial still forms a minor third having the strike note C_5? Was this distinction really made in literature? It seems not.

Otte wrote in 1858 that among the three most important partials of the bell - apparently he meant the hum note I-2, the third I-3 and the nominal I-4 - the middle one could be shifted by the ancient founders as they liked, and as the higher sounding bell required.[52] He supposed that for the peal of bells C_4-F_4, the C_4 bell had an F_4 instead of the usual E_4^{b}. And for two bells C_4-E_4, it was to him self-evident that the C_4 now had a major third E_4. In the most favourable case - if Otte was expressing more than wishful thinking - this demand for quality may have been based on faulty observations, because there is no doubt that a bell with the partials C_4-C_5-F_5-(II-3)-C_6 never has been made, and perhaps cannot be made. The reason is the tight structure of the first group of partials, a point we mentioned before. Perhaps it was for the same reason that Otte heard C_4-B_4^{b}-E_5^{b}-(II-3)-C_6, a bell in which the third indeed forms a fourth with the fundamental, although the distance between the strike note and minor third still is the latter interval. Apparently, there is much cause for confusion, as was already discerned by François Hemony.

When François Hemony in 1653 wrote to Athanasius Kircher about bells, one of his remarks was that there was an essential difference between his bells and the bells Kircher discussed in his famous Musurgia Universalis.[53] This difference, amongst others, was in the fact that completely different demands were made for carillons than for church bells. In fact this is one of the essential aspects in designing a major-third bell. Starting from the traditional minor bells it doesn't turn out to be that difficult - based on the idea that all partials may deviate considerably - to create a strongly raised minor third by using relatively small profile changes. And if that strongly rised minor third is within the ample standards of the major third, one could justifiably speak of a major bell. Indeed, this happened repeatedly, and in particular in the case of founders and musicians who could afford from their traditions, less stringent demands upon the tonal purity of their bells.

This holds, for instance, in Germany. At the bell symposium at Limburg an der Lahn on 6-9 June 1951, these standards for swinging bells were officially established. With respect to the strike note - for the sake of convenience we write one octave interval below the nominal I-4 - the limits were as follows:[54]

hum note	− 60 to + 20 cents
fundamental	− 35 to + 20 cents
minor/major third	− 25 to + 25 cents
fifth	a diminished fifth is acceptable for minor bells; nevertheless the perfect fifth is advisable. An augmented fifth is tolerable both for minor and major bells!
nominal	reference partial via strike note

These so-called Limburger theses have never been revoked in Germany. On the contrary, they have been confirmed again and again.[55] There was never any fundamental criticism. Even in 1986 on the part of German swinging-bell experts there was a sharp protest against the so-called overcultivated tonal purity, to confirm emphatically the above-mentioned standards again.[56] Truly, a very sharp contrast to the carillon standards developed, amongst others, in The Netherlands by the advisory committe of the Dutch Carillon Society, in which (in respect to the strike note, the nominal I-4) at least the hum note, fundamental and minor third have to be reached within 3 cents.

It will be clear that, based on the German guidelines, the realization of a bell resembling a major-third bell will be much easier than if an accuracy of a few cents is desired. A typical specimen of such a bell with visually only slightly aberrant profile curves is the following:

hum note	C_3 + 30 cents
fundamental	C_4 − 35 cents
major third	E_4 − 35 cents
fifth	G_4 − 100 cents
nominal	C_5 + 0 cent

These deviations are not small, but fall almost completely within the Limburger guidelines! On the other hand, there have always been musicians who were not deceived by this. They knew very well that such a bell (or a similar one) could give the impression of being a major-third bell, but in fact it was not.

An expert campanologist, the German Karl Walter, wrote in 1913, for example, that he had never came across a major-third bell that was also correct in the other partials. Only once did this result appear to have been reached, but upon closer listening, the hum note in that bell proved to be false. He expressed an opinion already formulated in Germany in 1896 by Johannes Bles-

52. Otte 1858, p. 55.
53. Kircher 1650, p. 519-523; Schott 1674, p. 357.

54. Beiträge zur Glockenkunde 1970 gives the official corrected "Richtlinien für die klangliche Beurteilung neuer Glocken", as established at the Deutsche Glockentagung of 6-9 June 1951. This is again described by Theo Fehn in his lecture (p. 39 ff.), and by W. Schildge (p. 93) etc.
55. Also in Ellerhorst 1957, p. 227 these "Limburger Richtlinien" - as they are called - are discussed.
56. Wagner 1986.

sing and almost at the same time in England by nobody less than Arthur Simpson. Much later, other experts like Christhard Mahrenholtz were to prove the same thing.[57] We can produce an almost endless list of names!

Major-third bells exist with strongly aberrant tonal structures, such as the so-called major bell from a foundry at Karlsruhe. Thanks to the fact that in the bell - changed in profile - a lower hum note and fundamental were accepted, the third could be raised by 2/3 of a semitone, giving the listener the impression of a major bell.[58] The case was the same in other foundries, such as the steel bell with approximately a major-third partial from the Bochumer Verein.[59] This bell was not a success, either. For similar reasons, a major-third bell which was developed by the Royal Eijsbouts bellfoundry at Asten in 1952, was not produced (Figure 3.6.1). This bell has, generally speaking, the tonal structure C_4-C_5-E_5-$G^{\flat}_5$-C_6, which does not really yield a beautiful sound.

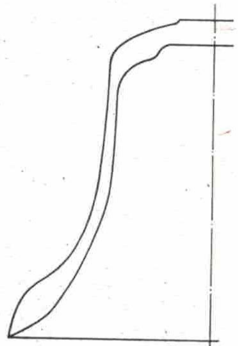

Figure 3.6.1. A major-third bell profile producing a diminished fifth.

In spite of this critical approach by expert listeners, also in Germany where attempts at the major-third bell were relatively frequent, interest in this type of bell continued. In 1950, it was stated at a bell symposium at Limburg an der Lahn that major bells are desirable, if only for the Maria antiphon, the Salve Regina motif. As said before, this is realized by the church bells C_4-E_4-G_4-A_4, and so the C_4 bell preferably should have a major-third partial E_4, and not the disturbing traditional minor-third partial $E^{\flat}_4$.[60] At the bell symposium at Neurenberg in 1956 this opinion was repeated, with the addition that one could at least ask for purity in tuning![61]

Finally, the problem of whether a major bell has ever been realized in the past can be approached in a different way. For if this is the case - deliberately in present times, or by accident in the past - all those claiming this, casually or convincingly, should be able to point out those bells and if possible produce sound analyses. Neither of the two has happened. Even the author of this study, who by virtue of his profession has viewed thousands of sound analyses of both historical and contemporary bells, has never come across a genuine major-third bell among them!

57. Walter 1913, p. 111-113; Blessing 1986, p. 69; Simpson 1896, p. 153; Hartmann 1897, p. 2; Mahrenholz 1948, p. 14.
58. Pamphlet of the Karlsruher Glockengiesserei.
59. Auslegeschrift 1.042.432 by the Bochumer Verein für Gussstahlfabrikation at Bochum (Germany) August 9 1957. This application was not continued, however.
60. On 7-9 November 1950 at Limburg an der Lahn, congress of the "Beratungsausschuss für das deutsche Glockenwesen". During the discussion of a lecture by Hans Rolli, "Zur Neufassung der Frankfurter Thesen".
61. Schildge in Beiträge 1970, p. 93.

4. DESIGNING BELLS BY EMPIRICAL RULES

4.1 Profile construction using arcs

Laypersons sometimes have the misconception that bellfounders develop their profiles only by means of computations. Nothing is less true. On the contrary, the founder designs his bell using a number of empirically established rules. Yet, this seems in contradiction with bell drawings found in literature.[62] There we see for instance a bell built from a number of arcs (Figure 4.1.1). Apparently, computations are necessary, the layman reasons.

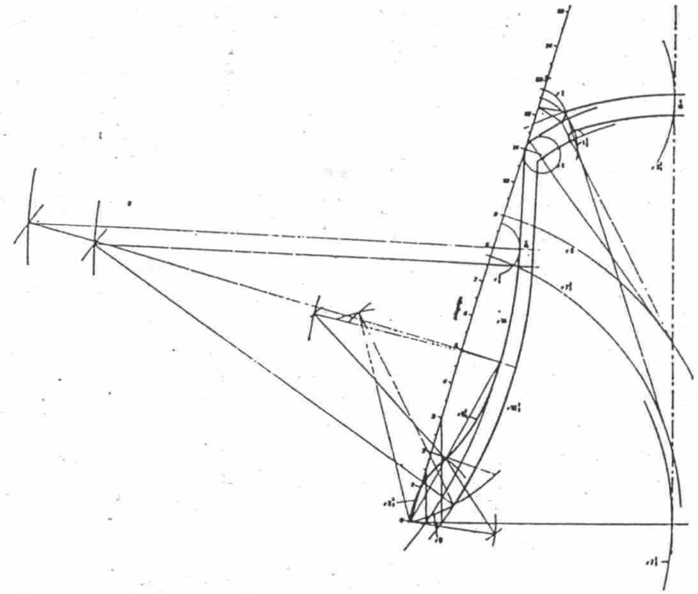

Figure 4.1.1. Bell profile of François Hemony (anno 1664) constructed with arcs of a circle and straight lines (after Lehr 1971, p.88).

In practice, however, what matters is a construction drawing consisting of some rules from which a bell of a particular model can be drawn to any size. The unit of length in these drawings is not a metric unit, but the stroke, which can be defined as, for instance, the 1/10 part of the largest diameter of the bell. All dimensions are then expressed in strokes, and for convenience the aim is to express their number as simply as possible using integers and fractions. Irrational numbers are to be avoided!

It is amusing to think that this linking together of circles shading off into one

62. Only some arbitrary examples out of many! Roujoux 1766, p. 94; Hahn 1802, tables I and II; Otte 1858, p. 63-65; Ellerhorst 1957, p. 56-67; Williams 1985, p. 180-182.

another strongly recalls the now often-used spline polynoms to express an arbitrary curve in several smoothly fitting functions! Even the spline, the mold for drawing curves, is a tool for bellfounders! There is a difference, as these arcs do not always fit as well as polynoms, which can be at least expected to have nodal points - points where the polynoms meet - in which not only the coordinates but also the first derivatives of both functions are equal.

Strictly speaking, a drawing rule using arcs is merely a mathematical translation of an already existing model. Therefore, some founders have used ellipses and not circles. This is certainly not illogical, as the upper waist of the bell is easier to describe by an ellipse than by several circles.

This gives an impression of physical considerations, as shown by an idea of the German father Johannes Blessing, an outstanding expert in his time, who at the end of the last century suggested composing the bell profile from parts of a hyperbola, a parabola and an ellipse.[63] He considered this approach much better than the - in his eyes arbitrary - linking together of arcs. But is there any fundamental difference? He had no answer to this question.

Even outsiders skilled in science were too often circumvented by those impressive, but nevertheless rather arbitrary arcs. The well-known French acoustician Marin Mersenne, for example, thought that he could recognize the frequency ratios of important musical intervals in the ratios between the radii of the various arcs.[64] This was only one attempt to establish that much-desired relation between profile and mode frequencies in easily recognizable and plausible looking computational rules. This was to be a very long road indeed!

4.2 Designing by empirical rules

On September 24 1653, the esquire Jacob van Eyck, municipal carillonneur of Utrecht, told the famous Dutch scientist Isaac Beeckman that the positions of the partials in a bell are determined by the bell profile.[65] Van Eyck added that the middle octave partial, the fundamental II-2, is often false, but he knew how to rectify this by making corrections to the profile.

It can be stated without any reservations that what Van Eyck put into words was the birth of a genuine scientific campanology. This approach alone made it possible for the brothers François and Pieter Hemony to found the first really pure sounding carillon in 1646, advised by Jacob van Eyck. This carillon was meant for the Wijnhuistoren in Zutphen (The Netherlands).[66]

The idea that the tonal structure is dependent on the profile is surely very old, as it cannot have escaped to the attention of any founder. However, a much deeper understanding was shown by François Hemony who - when asked - wrote in 1653 to nobody less than Athanasius Kircher that the ratio between the height of the bell and its largest diameter from early days is 12:15, but that it could also be 11:14.[67] But if, he added, that ratio is changed, it will affect the whole profile, because "one ratio depends on the other, and from the required ratios of width, height and thickness results that agreable resonance that is perceived in fine bells". It is a correct statement that was systematically overlooked in the later discussed ring theory (Section 5.4), and it also implies that it is not correct to suppose that every partial resides in a very particular part of the bell. This seems to be because of the fact that at a position where one partial has an antinode, another can have a node, or is only weakly heard there.

But which methodology is used in designing a bell? In general two approaches exist. In the first method a relation was sought, knowingly or not, between the timbre of the bell and its global profile. This method was especially used by founders who made only swinging bells that had to sound beautiful and sonorous, but that were not necessarily expected to have a clearly recognizable pitch with a consonant arrangement of partials. For that reason these bells have not to be tuned.[68] We will not occupy ourselves with the design of such bells in this publication, but discuss bells that should be designed very accurately according to a particular tonal structure, notably bells for a carillon.

The tradition that started with Van Eyck and was revived in our time by Simpson is based on an exact description of the relationship between the lowest five partials, precisely defined in tone, and an accurately described geometrical profile. Only then can a profile be designed with partials that together form the chord required, a chord that for a carillon is to be perceived as a closed unity at a certain pitch. Such a bell can be used in melodies. This may also demonstrate that this chord is to be realized exactly, and therefore the profile must be known equally exactly.

Practice has shown, however, since the 16th century, that a cast bell never exactly corresponds to the designed profile, as both the technique of modelling and casting introduce all kinds of changes that are difficult to predict. For that reason, carillon bells were, since the Hemony brothers, always cast somewhat thicker than would actually be necessary. After casting, the thickness of the bell was corrected to make the partials sound in the required structure. This procedure is known as the tuning of the bell.

There are quite a number of misunderstandings about this tuning. Some think that whatever the design, even a bell with a wrong profile can be corrected by tuning. Nothing is less true, however. The English Canon Arthur Simpson, who reinvented the art of tuning in the 1890's, was very correct in his remark that some bells are so unsatisfactory that they can only be sent back to the furnace.[69]

It must be stated emphatically that for a correct design not all relations be-

63. Blessing 1897, vol. 22, p. 27.
64. Mersenne 1636, Harmonicorum libri, p. 159 and Harmonie universelle, p. 36-37.
65. De Waard 1945, p. 310.
66. For more details on this fascinating period in history see Lehr 1959, p. 25-30 and Lehr 1971, p. 183-187. The Zutphen carillon was destroyed in 1920 by fire. One of the bells is left at the National Carillon Museum at Asten.

67. Schott 1674, p. 356-360.
68. Lehr 1986.
69. Simpson 1896, p. 152.

tween profile and partials have to be known. The reason is that the bellfounders of the past, if we assume that they knew which tonal structure a bell should have, could start searching in belfries for this ideal bell that perhaps had been founded before by colleagues, knowlingly or not. This assumption about such coincidences is proven by a remark of François Hemony, who had noticed this already in 1653.[70] And even now we can hear that in the Middle Ages some almost pure-sounding octave bells were founded.[71] In short, the founder could copy the basic design, and then add small changes as a personal touch, using his limited knowledge of the relation between overtone structure and profile.

Let us now return to the Hemonys and wonder how they, as original founders, were capable of designing and founding a pure-sounding bell on purpose. Unfortunately, we can only guess at the answer. It seems probable that they had searched for an existing bell, once they knew - thanks to Van Eyck - which bell chord was the ideal, and what bell-design corresponds best to this ideal, and then copied and adapted this bell. If this assumption is right, it would no longer seem possible that they created the ideal profile from scratch, using some universal method, or that they had so much knowledge of the relationship between profile and partials that they could transform any bell model, however bad sounding, to a good-sounding specimen. It is even more probable that they did not have such knowledge, as no source gives even the faintest hint to that answer. Such knowledge is not really necessary. In theory such an universal method is at least available today. Let us therefore consider the technique as it is used at the Royal Eijsbouts bellfoundry in Asten.

Two types of graphs are used in designing: graphs for changes in thickness and for changes in profile. The first type predicts how much a certain partial will change in tone if at a particular site and over a certain length, a certain amount of material is taken away, either on the inside or outside of the bell, reducing the thickness of the wall. Such graphs are shown in Figures 4.2.2-3. They are based on a division of the inside and outside into 32 numbered zones (Figure 4.2.1). Each zone has a length of 2.5% of the diameter. For the rest, in these particular graphs the changes are given in relative units, i.e. both in thickness and in tone. We find, for example, that the fundamental will fall strongly if the wall thickness is reduced on the inside in the sectors 24-32, or that the nominal will rise at the outer side in sector 1, etc.

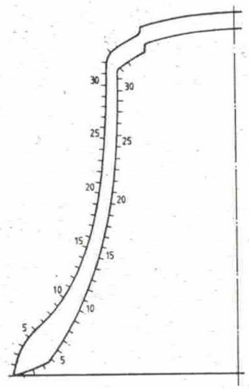

Figure 4.2.1. The division of the inside and the outside of a bell into 32 numbered zones.

These graphs, at first developed purely empirically, but nowadays also by

70. Schott 1674, p. 359.
71. Simpson 1896, p. 152

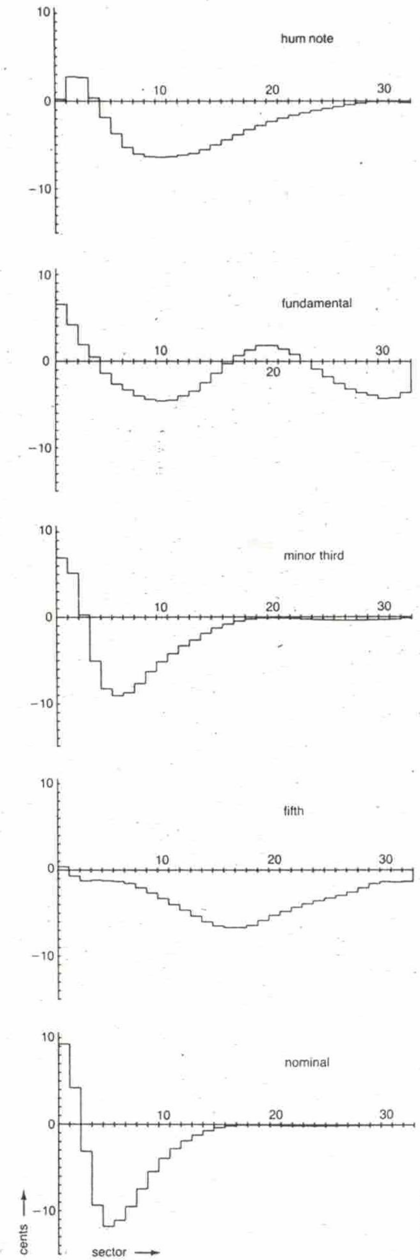

Figure 4.2.2. Graphs for changes in thickness on the inside of the bell.

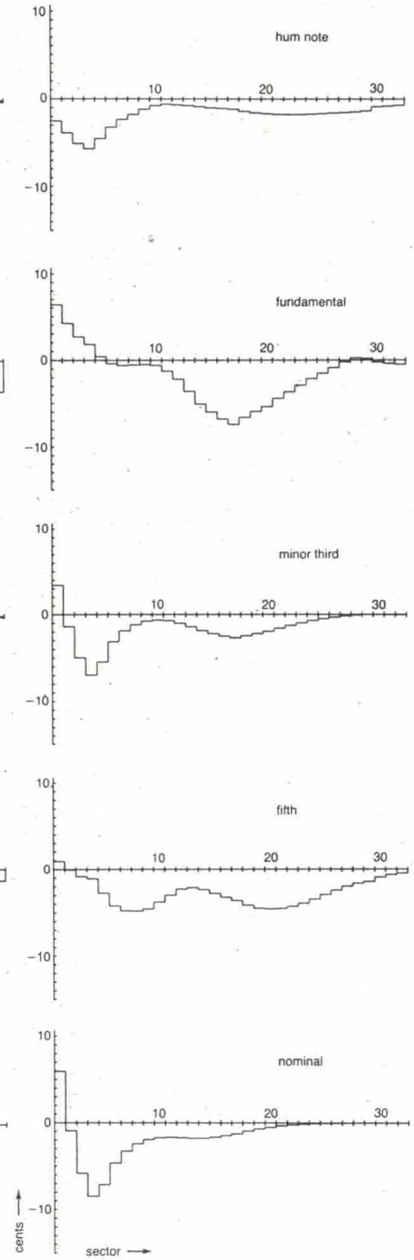

Figure 4.2.3. Graphs for changes in thickness on the outside of the bell.

computation,[72] can be used for tuning a bell (at least for the graphs for the inner side). Obviously, a bell can hardly be tuned on the outside because of its decorations and inscriptions. The graphs for the inside are, therefore, known as the tuning graphs.

In the graphs for thickness changes, it is assumed that in a certain sector the tonal change is proportional to the change in thickness. If twice as much bronze is removed, the tone will rise or fall twice as much. This linear relation is only valid for relatively small changes in thickness. When they become so large that they affect the essential profile, this simple relationship changes into a more complicated one. It is, for that reason, that the thickness graphs are only valuable within a small range. Let us now consider the profile graphs in Figure 4.2.4.

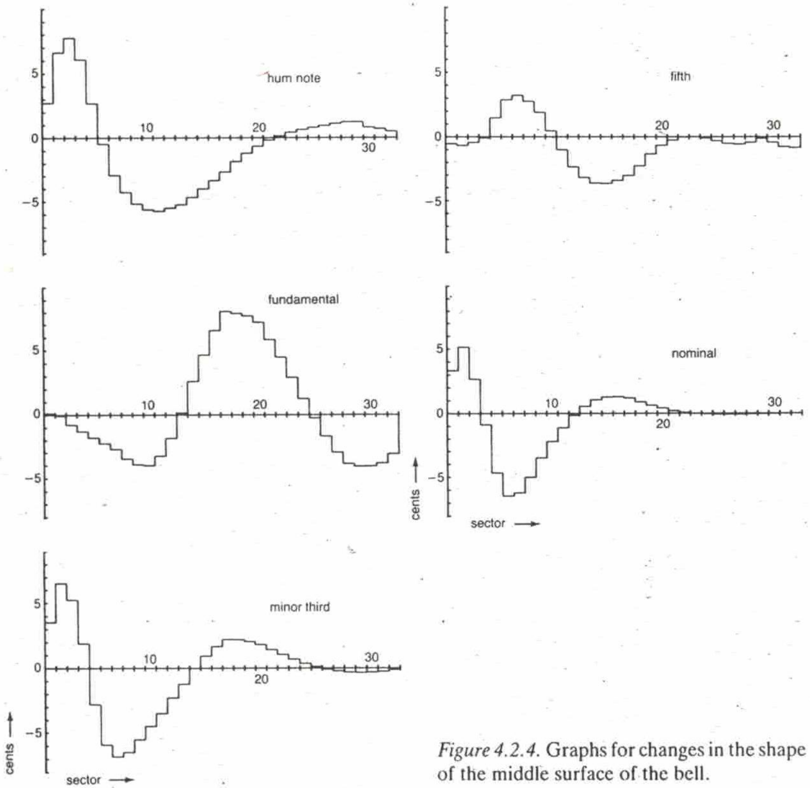

Figure 4.2.4. Graphs for changes in the shape of the middle surface of the bell.

Graphs for profile changes predict how much a partial will change if a certain sector is moved out over a certain distance. In practice, a single sector will never be moved in or out. One or more adjacent sectors on the left and right also have to be moved, of course, although increasingly less according to the distance from the central sector (Figure 4.2.5). In fact, this results in genuine profile changes, which explains the name of these graphs.

72. Van Heuven 1949, p. 82-86; Lehr 1965; Rossing 1984, p. 200. These investigations with the help of the computer were carried out at the Technical University of Eindhoven.

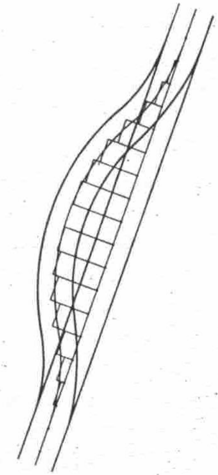

Figure 4.2.5. An example of changing the middle surface of a bell.

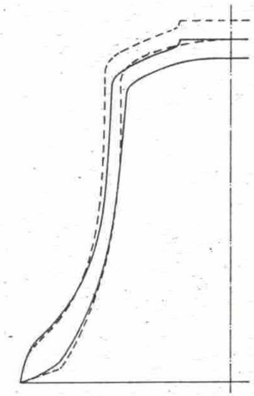

Figure 4.2.6. Two different bell profiles which give identical partials in the lower part of the sound spectrum.

As rather large profile changes are quickly introduced with these graphs, they soon lose their strict validity, and therefore they can only be used within a relatively small geometrical range. Nevertheless, they can be very valuable if the difference between the profiles of two octave bells from Figure 4.2.6 is under investigation. But, again, the early founders did not have this detailed knowledge, and, therefore, they probably stuck to what their eminent predecessors had achieved, keeping within the tradition of the carillon.

All this shows again how creative the Hemony brothers, together with Van Eyck, must have been in the seventeen forties. They not only formulated a bell chord that could be realized in practice, but they also were the first to define the required profile in such a way that pure sounding bells were no longer chance hits. And were their followers their imitators? For carillon founders, strictly bound to the octave bell, this might be true, while the founders of swinging bells were far less restricted. It was, therefore, especially in Germany, the country with its many heavy swinging bells, that research on profiles has continued to the present day.

Such investigations were not always successful, as is shown by the long search for the major-third bell. For if there is one problem which shows that the graphs for thickness and profile changes mentioned above are only valid within a relatively small range and that they can only be changed very cautiously, it is this search for the major-third bell. This bell was therefore not found in a conventional way, as will be discussed in detail in Chapter 6.

5. THE ANALYTICAL METHOD

5.1 Introduction

In 1644, when François Hemony, together with his brother Pieter and on the advice of Jacob van Eyck, succeeded in making the first really well tuned carillon, this was not only a milestone in the history of music, but also in science. The idea that beautiful profile lines were required for a beautiful sound was no longer current.[73] Yet, this revolutionary idea, which proved to be of decisive importance for the carillon founders, was at first rejected among scientists. The scientific investigators of those days stubbornly continued to search for a recognizable, and above all simple, arithmetical relationship between bell model and tonal ratios, due to the lack of any physical insight in the vibration patterns of a bell and/or an adequate mathematical calculus. Even if the founder was only interested in the result, a pure sound, and even if the question of the profile that guaranteed that superb tone eluded the founder, it was science that had to provide an answer.

Obviously, we can give several examples, such as Marin Mersenne who started from the erroneous premise that the bell has a major third of 5/4 in the lower part of its tonal spectrum. He explained this from the fact that the bell (excluding the canons) usually has a height of 0.8 times the diameter, and 1/0.8 is 5/4.[74] The same Mersenne thought, as was already pointed out in paragraph 4.1, that the various arcs from which the profile is composed, have the ratios of the partials in them. Is he to blame for this idea? Was not the 17th century a period in which interest for such problems had just awakened, and was it not the string and the organ pipe in which that much desired simplicity had been found, although empirically, at first? So why not in the bell? Only when the differential and integral calculus were definitively introduced in the eighteenth century, did the idea change, that vibrating media like a bell and gong would not reveal their secrets as easily as was hoped!

In the more popular literature this simplistic approach persists even until now, when the layman asks how the golden ratio is comprised in the bell. Even such a respectable author as Otte in the nineteenth century was a victim of this idea, as he thought that a bell profile can only be correct if the frequency ratio of every important interval is to be found in the ratio between radius and the height at certain sites of the profile.[75] And he was not alone in this opinion.

However one judges this somewhat too simplistic approach to the bell problem, it is in any case an analytical method. In its most elaborate and scientific form - and now we are in the 18th century - the aim of this method is to formulate an equation of motion for the vibrating medium, the solution of which gives the mode frequencies as a function of the geometry of the vibrating medium, i.e. the bell.

73. Lehr 1963, p. 88-118.
74. Mersenne 1636, Harmonie Universelle, p. 5.
75. Otte 1858, p. 64.

This equation is composed of one or more differential equations of second or higher order that have to be solved under certain boundary conditions. One of the first to formulate such an equation was the great mathematician Leonard Euler (1707-1783). Only in a very limited number of cases did this lead to success, at that time or later, and in even fewer cases, did this success lead to a solution in which the computed frequencies were related to the geometry of the vibrating body, thus confirming what was shown experimentally in a string, for example.

Euler came to the conclusion that the bell is actually too complicated to compute all its mode frequencies.[76] Later, at the end of the 19th century, Lord Rayleigh agreed.[77] This can still be confirmed, at least if this problem is to be solved strictly mathematically. Yet it was Rayleigh, as we mentioned in paragraph 3.1, who considerably simplified the bell by assuming that at least the lowest mode frequencies are pure inextensional vibrations, and thus the term for extensional energy in the expression for the potential energy can be neglected. Even if the middle surface could have been put into a formula, mathematics would fail to resolve this insoluble equation. Only for simple media, like a hemispherical bell with uniform and small thickness, were solutions possible.

Rayleigh's method had, at least for the bell, a rather severe disadvantage. By neglecting the extensional term, his outcome did not include any group above one (I). As we said before in paragraph 3.1, Love's approach in 1888 was more fundamental, though as a result of even larger complications, practical results would not emerge.[78] So it was a cul-de-sac.

Yet hope never disappeared completely. It is interesting that there has been at least one serious theory that, on the one hand, did not have this simplistic approach of relating frequencies to dimensions by simple arithmetical rules, but in which, on the other hand, complicated equations were avoided by experimentally tested computations. This is the so-called ring theory, an approach in which the bell is thought to be built up of linked rings. The most appealing variant of this method was to compute the mode frequencies for each ring and then use these together with experimental results to determine the frequencies of the combined rings. Thus, the questions of which mathematical function would describe the middle surface, and whether bell sounds result from purely inextensional vibrations were avoided. It is a model for computations that will be discussed in more detail in paragraphs 5.3-4, but we can already note here that this model failed completely!

5.2 Conditions for the generation of nodal circles

Lord Rayleigh assumed that at least the lowest partials in a bell are pure inextensional vibrations, thus vibrations for which the middle surface neither

76. Euler 1764.
77. Rayleigh 1890.
78. Love 1888.

stretches nor shrinks. In the previous paragraph we already pointed to the fact that this idea is wrong. If we neglect this aspect for a moment, then it is interesting to note that the consequences of purely inextensional vibrations on the presence of nodal lines, particularly, on nodal circles, can be computed on a purely mathematical basis. Again it was Rayleigh who was the first to do so in 1890, followed in 1909 by Abraham Vas Nunes in The Netherlands.[79]

The largest problem has already been noted: is it possible to express the middle surface of a bell in a mathematical expression? Rayleigh tried a one-bladed hyperboloid, covered by a circular plate, an approximation that, at least for a global description of the bell, is not bad at all. His final conclusion is that, at least on mathematical grounds, the mode frequencies with $s=2$ - two complete waves on the circumference - cannot have nodal circles. For all others - vibrations with $s=3$ or more - this is possible, but they would lie outside the bell surface! Remember that he did not compute frequencies.

The involvement of Vas Nunes, who repeated Rayleigh's computations with an identical result, had a curious history. The "Vereeniging voor Noord-Nederlandse Muziekgeschiedenis" offered a prize on May 1 1905, for an answer to the question:[80] "What are the scientific and physical foundations that made the bells of certain Dutch founders the best from an artistic point of view?" No answer was received for this question, and that is why Vas Nunes, on a proposal of his future thesis supervisor and Nobel laureate Pieter Zeeman (1865-1943), chose this problem as the starting point for his thesis. How much the computations of Vas Nunes ran aground on the mathematical intractability of the bell profile, and however many other problems had to be solved following that, the better were the results of his aural tests, listening to the carillon founded by François Hemony in 1656 for the Zuidertoren in Amsterdam.

5.3 The mode frequency according to the ring theory

When Leonard Euler in 1764 published a promising study titled "Tentamen de sono campanarum", it was - whatever the judgement of the results of his computations - the first strictly mathematical approach of the bell problem, though its title is somewhat deceiving.[81] Euler primarily discussed in this publication the mode frequencies of a circular ring, and he and his successors had good reasons to do so. The basis of this approach was to regard a bell as composed of rings, then to determine the formula for computing the frequencies of a ring, and finally to apply these to the bell according to a simple rule, assuming that a bell is just a pile of rings with various diameters and thickness.

This approach is, at least for the layman, far more obvious than Rayleigh's approach at the end of the 19th century, the first applications of which were described in the previous section. He considered the bell as a solid of revolution and not as a series of piled up rings. Rayleigh's idea was not quite brand new, as in 1789 Jacques Bernoulli (1759-1798), a member of a famous family of mathematicians, preferred to consider bell vibrations as vibrations in a plane that is composed of innumerable little planes, each of them thought to be composed of two perpendicularly crossing bent bars.[82] And later it was Chladni who stated in 1802 that a bell is fundamentally a bent plate, thus strongly emphasizing its three-dimensional character.[83] But let us return to Euler.

Euler made a mistake in neglecting the tangential movements along the ring surface. By doing so, it was not correct to assume that a sinusoidal vibration could be generated around the circumference, and so his conclusion was also incorrect. The mode frequencies of a ring are not proportional to the square root of $s^2(s^2-1)$, in which s is the number (2 or more) of standing waves around the circumference. As we are in principle only interested in the frequency ratios that determine the intervals between the various partials, we set the hum note frequency equal to one. The following series is then obtained:

value of s:	2	3	4	5	6	etc.
ratios:	1 :	2.4495 :	4.4722 :	7.0711 :	10.2470	etc.
in cents:	0	1551	2593	3386	4029	etc.
tones:	C_4	E_5^b/E_5	D_6	B_6^b	E_7+	etc.

Euler's pupil Golovin, who dedicated himself in 1781 to computing the mode frequencies of hemispherical bells from the successful glass harmonica - thereby starting from a ring - was even more rigorous in simplifying the equation of motion by neglecting a certain term.[84] His motivation has not become clear. It is certain, however, that his results are of little use. He computed that the frequencies are proportional to $(s-1)^2$, yielding:

value of s:	2	3	4	5	6	etc.
ratios:	1 :	4 :	9 :	16 :	25	etc.
in cents:	0	2400	3804	4800	5573	etc.
tones:	C_4	C_6	D_7	C_8	$G_8^{\#}-$	etc.

One year later Euler was to reach the same conclusion for a bent and closed ring, thus departing from his earlier computations in 1764. He also noted that the overtones quickly became so high that they could no longer be heard, and therefore the ring appeared to have an empty sound. In his opinion this gave a pleasant musical effect. He also thought that the same phenomenon occurred for round discs and bell-shaped shells. As long as this remark does not refer to real bells, which are thick shells with varying thickness, this is a more or less valid conclusion.

79. Rayleigh 1890; Vas Nunes 1909, p. 77-93. The same computations can be found in Bouasse 1927, p. 389-390.
80. Tijdschrift der Vereeniging voor Noord-Nederlandse Muziekgeschiedenis, vol. 8, 1908, p. 178-180.
81. Bernoulli 1786.

82. Chladni 1802, p. 192.
83. Euler 1764. Also see Todhunter 1886, p. 55-57; Truesdall 1960, p. 142-146 and p. 320-322.
84. Golovin 1781.

The acoustician Ernst Chladni also worked on the frequencies of a ring.[85] He concluded that these frequencies were proportional to $(2s-1)^2$, which yielded an arithmetically beautiful series with squares of 3, 5, 7, 9 etc.

value of s:	2	3	4	5	6	etc.
ratios:	1 :	2.7778 :	5.4444 :	9 :	13.4444	etc.
in cents:	0	1769	2934	3804	4499	etc.
tones:	C_4	$F^{\#}_5$	F_6+	D_7	A_7	etc.

It is interesting to determine which investigator was closest to the truth. We know that the frequencies of a ring with a rectangular cross-section are determined by the square root of $s^2(s^2-1)^2/(s^2+1)$.[86] This gives the following values:

value of s:	2	3	4	5	6	etc.
ratios:	1 :	2.8284 :	5.4233 :	8.7706 :	12.8663	etc.
in cents:	0	1800	2927	3759	4423	etc.
tones:	C_4	$F^{\#}_5$	F_6+	$C^{\#}_7/D_7$	$G^{\#}_7$+	etc.

So Chladni was very close to the real value, while, on the other hand, we may wonder why Euler and others apparently never listened to the overtones of a ring, a very simple experiment indeed!

From this ring, the switch to the bell had to be made. However, what all authors considered the goal of their computations was never realized. At first Euler, for instance, came no further than his remark that whatever was the height of the ring, this had no effect on the computed frequencies. It was - if the ring remained purely cylindrical - a correct but incomplete observation, as more mode frequencies exist in such a cylinder than only those without node circles. Later researches such as Pfnor, who in 1848 investigated the Hemony carillon in Darmstadt - destroyed in 1943 - had understood this very well.[87]

Later on Euler reverted from his original idea of considering a cylinder merely as a ring with a particular height, for, as he finally said, it could be that vibrations in a bell are subject to quite other dynamic rules. In his view, a totally new theory had to be developed for computing mode frequencies in shells of any form. The method that had been followed until then was only meant for strings and other easily describable vibrating bodies with constant thickness.

On similar grounds others did not agree with the ring theory, either. Chladni, for example, rejected this theory, as it forced the assumption that each partial resulted from a particular ring of the bell, which was contradictory to the facts. Blessing noted much later that the nodal circles still would be unexplained.[88] Otte observed in 1858, referring to the medieval Vincent de Beauvais, the very same vibration is present over the whole wall of the bell,[89] so it does not reside in a particular ring.

85. Chladni 1802, p. 198.
86. Rayleigh 1877, par. 232-233; Bouasse 1927, p. 394-396; Timoshenko 1928, p. 408-411.
87. Pfnor 1848, p. 336-337.
88. Blessing 1895, vol. 20, p. 100.

As the ring theory failed as a pure theory, there was a search for formulas along experimental paths. Chladni developed an arithmetic relation for the mode frequencies of a thin hemispherical bell from the glass harmonica, a formula arising from campanological literature again and again, even nowadays.[90] This formula expresses, amongst others, that the partial frequencies of such a bell are proportional to s^2.

value of s:	2	3	4	5	6	etc.
ratios:	1 :	2.2500 :	4.0000 :	6.2500 :	9.0000	etc.
in cents:	0	1404	2400	3173	3804	etc.
tones:	C_4	D_5	C_6	$G^{\#}_6$–	D_7	etc.

Chladni was aware of the fact that a varying thickness did affect the frequency ratios. But he did not notice, although it was the logical consequence of his remark, the fact that the overtones of the bell from the glass harmonica he described are only partials of the first group, albeit with the mode function of the hum note of a swinging bell, therefore without a nodal circle. Apparently we must assume - and experiments show - that if the model of the bell becomes geometrically more complex, a nodal circle also arises for the first group. This circle, however, is not in the sound bow, as for the tones of the second group, but in the waist (Figure 3.3.1).

In 1865, Helmholtz was to express the same opinion as Chladni.[91] He, too, realized that a varying thickness of the wall must have a notable effect on the position of the partials. He went even further by adding that it was apparently possible to determine empirically a bell model for which the overtones have harmonic ratios, as if the 17th and 18th century carillon founders in The Netherlands never existed!

It was the American Kalnins in 1964 who contested the unavoidable failure of the analytical method by using a computer.[92] He divided a geometrically complicated axially symmetrical shell into segments of suitable height, ending with a number of cylinders, truncated cones or other easily describable elements. Each element can be computed and, moreover, can be linked to the others by choosing the right boundary conditions. It will be clear that this approach, in itself not very revolutionary, was only applicable once computers were available.

Evidently, the method of Kalnins was in fact another ring theory, but with the important differences that frequencies were not computed for every ring mode, but bending forces and shearing strains were considered. Thus, what was never clearly stated in the past was now no longer under discussion: surely

89. Otte 1858, p. 56.
90. Chladni 1802, p. 197. Recently, for instance, the formula was nominated by Schwartz 1984, p. 6, as representative for the bell.
91. Helmholtz 1865, p. 125, idem (English translation 1954), p. 72-73.
92. Kalnins 1964.

in the ring theory one can use hypothetical rings, if their coupling together is done correctly.

It would have been worthwhile to apply this method to bells as well, were it not that at the Technical University of Eindhoven in 1972 the so-called finite element method was introduced for this time-honoured belfry instrument.[93] Suddenly earlier mathematical and computational problems, fundamentally insoluble, disappeared. In Chapter 6 we will return to this subject. Here we can say that again a ring theory was involved.

5.4 Ring theory and profile design

In the previous section we were faced with the failure to determine the mode frequencies of a bell using those computed for rings, as these attempts did not go much beyond the ring itself. Nevertheless, ring theory was indeed applied by some founders in designing their bells. Here, we have left the realm of strict science, however, in order to enter a twilight zone in which all kinds of bogus scientific speculations were applied to bells without much self-criticism. So the simplistic idea slipped in, without any theoretical foundation, that the ratio between the frequencies of two partials should be reflected in a ratio between two dimensions. Generally, the idea is that this ratio should occur, for example, in the ratio of the thickness of the rings at two different heights on the bell profile, or in the ratio between the radii at these same heights, etc.

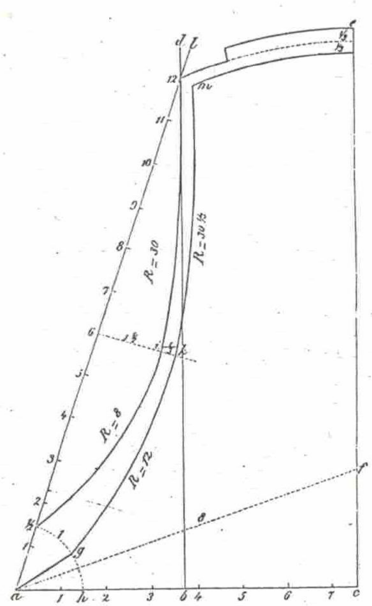

Figure 5.4.1. Construction of a bell profile (after Otte 1858, p.64).

It was the often-mentioned Otte amongst others, who in 1858 presented the profile construction of Figure 5.4.1.[94] The unit in this drawing is the stroke, in this case defined as the 1/15 part of the diameter. At point 12 of the stroke line the diameter of the bell is half, i.e. 7.5 stroke. The reasoning goes as follows: If at a particular height of the outer profile

12/oblique height = 15/radius,

then a partial can be heard in that ring that corresponds to this ratio. Thus he determined that at 9.6 strokes from the shoulder the diameter is 12 strokes. And as 12/9.6 = 15/12 = 5/4, a major-third overtone should reside at that site. Or, to give two other examples, at 9 strokes from the shoulder the diameter is 11.25. And as 12/9 = 15/11.25 = 4/3, a fourth resides at that place; at 8 strokes from the shoulder the diameter is 10 and a fifth of 3/2 can be heard there.

93. Banens 1972.
94. Otte 1858, p. 62-65.

It should be clear that such reasoning is so unrealistic, that no further discussion is necessary. In fact, Otte undermined his own thesis by presenting a profile construction in which these diameters were slightly different than he had visually measured. If he had computed the various dimensions, he could have found that these ratios were not as perfect as he had measured in the drawing. In passing he noted that where the height is 4/5 of the diameter, a major third should be present in the bell sound; because the upper ring has a diameter of 7.5 strokes, and the lowest 15, in that ring an octave (2/1) should be heard. Otte should have listened to his bell in reality. Assuming that he was in the slightest bit musical, he would have noticed that all these harmonic intervals were absent, no sign of a major-third partial being present. No, this paper "bell" only sounded in Father Otte's vicarage, and nowhere else!

Otte certainly was no exception. The German Wrede, for instance, in 1909 started from the idea that the thickness of the ring immediately under the shoulder determined the position of a particular partial.[95] No word about the other overtones! The gist of his reasoning is that when the ratio of thickness of the upper waist to the thickness of the sound bow equals 1/2, the octave of the hum note can be heard there. If this ratio is 2/5, then it is a diminished sixth (8/5); if 3/8, a fifth (3/2); if 1/3, a fourth (4/3); if 5/16, a major-third (5/4), and if 1/4, the hum note again (1/1). In the latter case, the hum note was to be heard both in the sound bow and at the upper waist. It will be clear that Wrede, too, never put his observations to the test of reality, nor could he give them a theoretical foundation. Yet both Otte and Wrede were representative of this period. If it were not for them, this period could be omitted from the history of campanology.

Of course, the fact that they were not founders plays a role in this. The English bellfounder Llewellins arrived at more sensible observations. His idea in 1879 was that the modes of vibration of the most important bell partials - apparently he only speaks of those partials that have no nodal circle in the sound bow - are identical around the circumference to those of the ring, but with a significant difference.[96] This difference is not to be found in the frequencies, but in the timbre. He wrote that, though a bell has the same type of overtones as a ring, the bell should have a model such that these overtones can be heard, but the hum note below them has to dominate. If the bell is too thin, the tendency towards splitting up the sound is so strong that the overtones become too loud. A bell that is too thick has only a hum note, which is also unpleasant. It reminds one not only of the qualifications German founders gave to their "leichte und schwere Rippen", but also of a remark of the German phi-

95. Neumann 1950, p. 51.
96. Anonymous 1858 (Llewellins), p. 1-15 and p. 25.

losopher Hegel at the beginning of the last century about the sound of the bell.[97]

So Llewellins sees a clear relation between sound and thickness, a relationship that was no doubt found in practise, but in his publication was given a quasi-scientific foundation. Nevertheless, the common sense of this founder finally won out, for, as he wrote, the important curvature and other details of the profile can only be established by experimentation. Unfortunately, even Llewellins did not succeed.

The last representative of this school was Neumann from Austria.[98] Although the first part of his thesis, published in 1950 (sic!), is a discourse in which the mode frequencies of a ring are computed with the help of a strickly mathematical bending theory, in the second part he loses himself in strange speculations. Here the basic idea was also that the frequency ratios between the partials should be reflected in the dimensions by a simple rule.

Neumann starts with a known profile in which he wants to demonstrate his hypothesis. Based on his theory of the vibrating ring he calculates the mode frequencies of a bell ring at an arbitrary height. Although these ring frequencies cannot correspond to those of the partials in the bell, by holding the inner profile in its original shape, he demands that the outer profile of the bell should be modelled in such a way that in every other ring of the bell the same ring frequencies would be present. His hypothesis says that the computed outer profile will be identical with the real outer profile.

At an arbritary point in the profile, the inner radius is r_1 and the outer radius r_2. By means of these measurements the wall thickness $d_0=r_2-r_1$, and the radius of the middle surface $r_0=(r_2+r_1)/2$ can be calculated. For a ring, the mode frequency f_s of partial s according to the bending theory is:

$$f_s = c_s \cdot \frac{d_0}{r_0^2}$$

in which c_s is a constant which is known from theory. In contrast with the elementary bending theory, c_s depends not only on s but also on the ratio r_2/r_1. This means that the interval between two partials is a function of this ratio.

The formula can be rewritten as:

$$f_s = 4c_s \cdot \frac{r_2-r_1}{(r_2+r_1)^2}$$

With the help of this formula, Neumann demanded for any other ring of the bell the same frequencies f_s. Holding r_1 at its original value on each height, he computed r_2 for each point of the outer profile.

According to his theory, the computed outer profile must be identical for each mode s. In reality this was certainly not true and, moreover, a fairly acceptable correspondence with the real bell was only actually found for s=2 of the first group, the hum note. So it was a cul-de-sac, as could be expected, and

97. Hegel 1835, p. 70.
98. Neumann 1950, p. 59-63.

Neumann in fact just made an attempt without giving it any theoretical basis.

Neumann was the last of a line of mathematicians who tried to compute bell frequencies by vague assumptions. More than twenty years later, in 1972, a completely new approach was to be followed at the Technical University of Eindhoven, a very successful one, indeed.[99]

99. Banens 1972.

6. THE NUMERICAL METHOD

6.1 General description

The previous chapter discussed the analytical method for computing mode frequencies in bell-like bodies. This method could be successful only if two conditions were met. First, the geometry of the bell-like body should be expressed in a mathematical formula. This is impossible for a bell, and therefore simplifications were introduced, such as that of Rayleigh who considered the bell as a one-bladed hyperboloid with a constant small thickness. It is clear, as any founder will confirm, that such a bell does not resemble a real bell musically. In such a bell, the partials would have completely different positions in the sound spectrum!

The second condition is that if the simplified bell actually can be defined by a mathematical formula, the resulting equation of motion for the vibrating bell still has to be solved with appropriate boundary conditions. But this also usually proved to be an illusion for these partial differential equations. And to think that this equation had already been simplified from another assumption - as we remember Rayleigh thinking - that the lowest partials result from pure inextensional vibrations. This implied in advance that the overtones of the second and higher groups were excluded from computations. But these groups also contain important partials, such as the fundamental II-2 and the fifth II-3. In conclusion, the analytical approach was a cul-de-sac for more complex vibrating bodies.

For some decades these problems could be evaded by applying numerical methods. The basic idea is very simple, as can be illustrated by an example in which both an analytical and a numerical method can be used. Suppose we want to know the surface of a certain area enclosed by part of a parabola. Using the analytical method, the formula for the parabola is integrated between the required limits. But this parabola can also be drawn on graph paper to count the number of square millimeters. As the parabola crosses many little squares at the border of the area to be measured, it will be clear that the numerical method, in fact, is an approximate method.

Someone who does not know the formula of a parabola or is not able to integrate it, will start counting, for a long time if necessary. Of course this has its practical limits, and an automatic counter, the computer, is necessary. It counts the squares at high speed after we have entered the line of the parabola as a limiting condition. At the same time it is clear that comprehensive numerical problems imply computations that can no longer be made by hand, however simple the steps might be, and could therefore only be solved when computers became available.

Wherever the analytical method fails, the numerical method seems to be the obvious way, if it is applied to the problem under survey. For bells, no squares have to be counted, of course. Here the problem can mainly be reduced to the interfering forces in the vibrating medium. That is why the finite element method was used. This method was applied to bells as early as 1972,[100] although the first published results appeared in 1983.[101]

The finite element method was introduced in the aircraft industry in the fifties when the analytical method had failed, and the processing of large amounts of numbers became possible by using computers.[102] It became possible to divide the model to be examined into a large number of geometrical elements. Each segment is described by a set of linear equations, the solution of which is fairly straightforward, thanks to the computer. So, this was done for the bell too!

It is obvious that for a bell with rotational symmetry, the elements are rings with simple cross-sections, such as triangles or quadrangles. On the one hand the dynamics of the rings are sufficiently known and manageable, so no elements have to be formed around the circumference of the bell; on the other hand it was in the vertical direction, along the mathematically complicated profile, that the analytical approach failed. It is amusing to note that the ring theory had returned, now in a correct form in which the inertial and tensial properties of all elements were united to a set of soluble equations. Ring frequencies were no longer computed, which of course is a very essential - and therefore necessary - difference in comparison with the old method!

In this respect, it is interesting to note that the finite element method, in fact, has had a long history, as it was certainly not only for the bell that a body to be computed - in whatever property - was divided into elements; this was done around 1900. Though this method led to solutions for less complex bodies, it did not for the bell. Perhaps because of this, the door was opened for various attempts with bells that could not stand the test of scientific criticism.

The application of the finite element method for the computation of the mode frequencies of a bell not only had the advantage that these could be determined accurately, but also that all partials could be computed if desired, whatever the group they belong to. This also removed the restriction introduced by Rayleigh at the end of the 19th century, which was contested by Love. The computations did not compel the restriction to pure inextensional vibrations, but they allowed, in principle, all types of vibrations.[103]

6.2 The computations of the major-third bell

In the previous paragraph we described how the mode frequencies of a bell can be computed numerically, starting from a known profile and using the finite element method. This is not revolutionary in practice, however, as a founder could - ignoring time and cost - found a bell according to that profile, to obtain

100. Banens 1972.
101. Perrin and Charnley 1983
102. See e.g. Livesley 1983.
103. Here we disregard that pure extensional vibrations, usually high in the sound spectrum, were until now not considered and therefore excluded from computations.

the same result as the computer. The reverse question is more difficult. As yet this question was restricted to the frequencies of the partials and could not include their volumes and decay times. Apart from this, what was more obvious than aiming that question at the major-third bell - until then unknown - thus the question which profile would produce such a bell. This was asked in 1983 by the Royal Eijsbouts bellfoundry at Asten, to the Institute of Fundamental Mechanical Engineering of the Technical University at Eindhoven.[104]

For the investigation of the major-third bell to have a reasonable chance of success, particularly in view of the many failures in the past, there was only one compelling demand for the future major-third bell: the minor third had to be changed to a major third, while the hum note, the fundamental, the fifth and the nominal had to remain the same. But it is also clear that a bell is determined by many more characteristics, and shifting a partial will not leave these undisturbed. Furthermore, it was desired that the new bell should not differ too much visually from the traditional model. In short, the degrees of freedom for the future major-third bell were chosen to be as wide as possible. Based on these starting points it could be expected that the first design of the major-third bell would differ from the minor-third bell, not only in the position of the third, but also in its other properties, in particular:

1. the position of the other partials, higher than the nominal I-4.
2. the sonority of the total bell sound, in which the volumes and decay times of the partials play a decisive role.

But how was the investigation to be carried out precisely? Two approaches were possible in principle. The first was to compute, using the finite element method, the partials of an existing profile that already inclined toward a major-third bell, and also the effects of small local changes in the particular profile in model and thickness, that is, the graphs for changes in profile and thickness. The direction of the profile changes to be made on behalf of a major third could be chosen using these results, then the mode frequencies of this adjusted model could be computed, including the profile and thickness graphs. This could be followed by a selection of another direction of change, if necessary, and this process could be repeated until the required model was found.

This so-called iterative method of optimalization has quite severe disadvantages, of course. There is no essential difference with the old method of the bellfounder, namely determining the new profile using thickness and profile graphs, except for two important but not essential exceptions. First, this method computes these graphs over and over again, so they apply very precisely to the model last tested. Secondly, the founding of test models is no longer necessary as they are "founded" and tested by computer. Nevertheless it remains a very complex procedure for which success is not certain at all. So, a second method was chosen.

In the starting profile, ten so-called design variables were chosen: five radii and five thicknesses (Figure 6.2.1). Together these form a set of fixed points for the inner and outer profiles. They were linked together by fluent curves using third-grade spline polynoms. The head of the bell was considered to be invariable. At first the design variables were varied at random, which in the computed sound spectra gave no useful results whatsoever. Then a more systematic search was started, and finally it was shown that if radius r_2 is increased, that is to say the wall is moved outside and the bell forms a "bulge", the third will rise. The other partials will also change - especially the fundamental which starts to rise - but such effects can be corrected by making the bell longer. Thus a first and very global design for the major-third bell originated. Using hindsight, this result was rather predictable, as this tendency is very clear from the graph of the third in Figure 4.2.3. But this graph was not known in sufficient detail at the time of the investigations.

104. Van Asperen 1984; Maas 1985; Schoofs a.o. 1986; Schoofs 1987

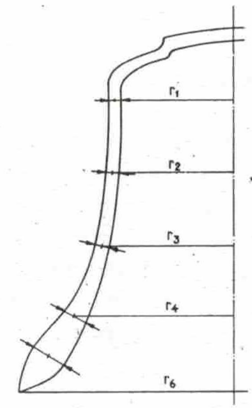

Figure 6.2.1. A bell with design variables.

The procedure followed until now may not have been essentially different from the use of empirical rules in the form of thickness and profile graphs or the method of iterative optimalization sketched out above. The final step to go from a global model to the exact model was fundamentally different. It used a so-called numerical experimental design, the basic idea of which is as follows:

Around the very global model for the major-third bell a geometric area is chosen, based on experience and assumptions, in which the exact model is expected to be. In this area a large number of models is chosen according to a particular systematic. In this case there were no less than 128 models, for which the mode frequencies are computed using the finite element method. Then, using the least squares method for each partial, the best-fitting polynom for the relationship between all these profiles and partials was computed. So each polynom describes the shift in the position of a partial as a function of the design variables. By March 1985, the exact model could be computed using these formulas. Although in fact it is an approximate method, in practice it yields very accurate results.[1] Figure 6.2.2 gives the result.

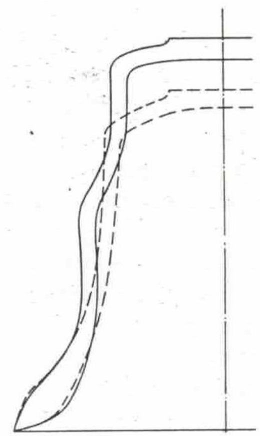

Figure 6.2.2. The first major-third bell profile, in comparison with the traditional minor-third bell.

6.3 The realization of the major-third bell in practice

When the first-computed major-third bell was founded, it turned out to have the tonal structure given in the third line of Table 6.3.1. In this table, intervals smaller than a semitone have been omitted to make it more easily readable. The traditional minor bell is given in the fourth line, arranged in such a way that the partials with the same vertical vibration pattern and the same number of meridian nodes are in the same column as the major bell. In the second line the global strengths of the partials after striking the bell are given.

partial	I-2	II-2	I-3	II-3	III-2	I-4	III-3	II-4	I-5	III-4	II-5	I-6
loudness	mf	f	f	mp	p	fff	p	mp	f	p	p	f
major bell	C_4	C_5	E_5	G_5	B^b_5	C_6	$C^\#_6$	E_6	G_6	A^b_6	B^b_6	C_7
minor bell	C_4	C_5	E^b_5	G_5	F_6	C_6	F_6	E_6	G_6	A_6	B_6	C_7

Table 6.3.1. Partial tones of the first major-third bell in comparison with the minor-third bell.

At the first striking of the bell, it was surprising that the partials III-2 (B^b_5) and III-3 ($C^\#_6$) put a clear stamp on the sound of the bell. In spite of the fact that these partials are rather weak, because the bell is struck near a nodal circle in the sound bow, they are definitely perceived. This is especially true for $C^\#_6$, as it could easily interfere with the nominal, with beats so fast that its presence in the sound was especially emphasized. Moreover, it was found that, even apart from the major-third partial, and probably even apart from the partials B^b_5 and $C^\#_6$, the timbre was rather different from the usual minor bell. Therefore, it seemed necessary to change the computed model of the major bell so that its timbre would not be too far away from the minor bell.

In order to avoid re-computation using the finite element method, these desired changes were carried out using the empirical rules mentioned before. A new rule was added to these in the meantime, namely how changes in overtones result from bulges in the middle of the waist. Moreover, it was soon found that only this bulge could transform the minor-third partial into a major third. Obviously, this is not the place to describe these experiments in great detail. It should be sufficient to state that the final major-third profile had a somewhat larger bulge than was at first computed, and also that the ratio between the thickness of the sound bow and the thickness of the waist was made larger, and therefore more nearly corresponding to the ratio in a minor bell.

As a result of all this, the bell not only had a somewhat darker timbre, but it was also possible to move the disturbing $C^\#_6$ partial to the octave partial C_6. Thus Table 6.3.2 resulted, in which the arithmetical ratios both for the major and minor bell are given.

Study of this table will show that the major bell has gained a harmonic seventh B^b_5- (7/4), which is hardly heard, however, as the bell is struck near nodal circle for that partial. Nevertheless, it forms - and this is another remarkable fact - together with the other partials, a harmonic series (1:2:3 etc.) which is much bet-

partial	I-2	II-2	I-3	II-3	III-2	I-4	III-3	II-4	I-5	III-4	II-5	I-6
loudness	mf	f	f	mp	p	fff	p	mp	ff	p	p	f
major bell	C_4	C_5	E_5	G_5	B^b_5-	C_6	C_6	E_6	G_6-	A^b_6+	B^b_6	C_7
	2 :	4 :	5 :	6 :	7 :	8 :	8 :	10 :	11.8 :	13 :	14 :	16
minor bell	C_4	C_5	E^b_5	G_5	F_6-	C_6	F_6+	E_6	G_6	A_6-	B_6	C_7+
	5 :	10 :	12 :	15 :	26.4 :	20 :	26.8 :	25 :	30 :	33 :	36 :	41

Table 6.3.2. The improved first major-third bell.

ter achieved than in the minor bell, as is also expressed by the easily perceptable difference tone between the fundamental C_5 and the third E^b_5 (6/5) or E_5 (5/4). It is easy to compute that this tone for the minor bell is a major tenth A^b_2 under the hum note C_4, whereas for the major bell it is the pleasant lower octave C_3.

Regarding the sonority, facts are less evident than for the harmonic structure. In fact, during the whole course of the problem there was a constant question whether a major bell should be a bell with a major-third partial and for the rest a bell with the traditional timbre, or a bell with a major-third partial having a timbre of its own. Orginally, the first type was sought, a bell having not only a major third but also the long-lasting sonorous sound of the minor bell. However, this proved to be impracticable, at least according to the empirical rules of the bellfounder. What was the case?

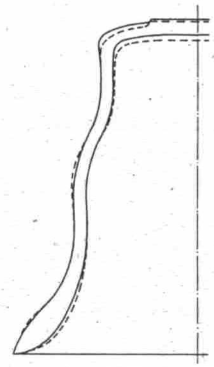

Figure 6.3.1. The improved major-third bell (dashed line) in comparison with the first computed major-third bell.

Having found some test models, it seemed better, in order to suppress the harmonic seventh B^b_5-, to make the wall of the major bell thicker than that of the minor bell. Quantitatively, this can be expressed in the so-called f·D value, the product of one-fourth the frequency of the nominal and the diameter in meters (section 3.1). For large minor bells this value is about 200 m/s; for major bells a value of 220 m/s seemed most suited. But this certainly had an effect on the sonority. The higher the f·D value - so for an invariable frequency the larger the diameter - the thicker the bell had to be to realize that frequency. But this also implies that the radiating surface becomes larger with increasing f·D value, thus increasing the acoustical damping and shortening the decay times of the partials.[105]

Apart from these things, the major bell also has greater internal damping than normal. At least, this seems plausible if it is realized that a bulge in the

105. Van Heuven 1949, p. 93-109; Lehr 1986.

waist of the bell is not exactly conducive to easy vibration of the waist. In short, this also makes the reverberation time shorter. But there is still more. It is a well-known fact that a bell with large material and acoustical damping is difficult to bring into vibration, and therefore cannot reach the sound volume which is possible for a bell with low damping, especially the minor bell. So, compared to its much older relative, the major bell is less sonorous and briefer in sound. It is doubtful, however, if this is a real objection in a carillon bell. We will return to that subject later. Here we will simply refer to Figure 6.3.1 in which the first profile of a major bell has been drawn.It should be noted that the Royal Eijsbouts has already applied for a patent on the major bell.

7. MUSICAL IMPLICATIONS OF THE MAJOR-THIRD BELL

7.1 The major-third bell in the practice of music

Having finished the definite design of a major bell, a complete carillon could be realized (Figure 7.1.1). Ample discussions led to a carillon of four octaves with the strike tones B^{b}_{4}-C_5-D_5-chromatic-B^{b}_{8} that are connected to the keyboard as C_5-D_5-E_5-chromatic-C_9. The lowest bell has a diameter of 945 mm (37 3/16") and a weight of 585 kg (1290 lbs). For the smallest bell these values are 151 mm (5 15/16") and 6 kg (13 lbs) respectively. The total weight of the bells is 3026 kg (6671 lbs). This first major carillon was made mobile, so it could be transported to as many places as possible to be heard by as many listeners as possible. And what was the opinion of these listeners?

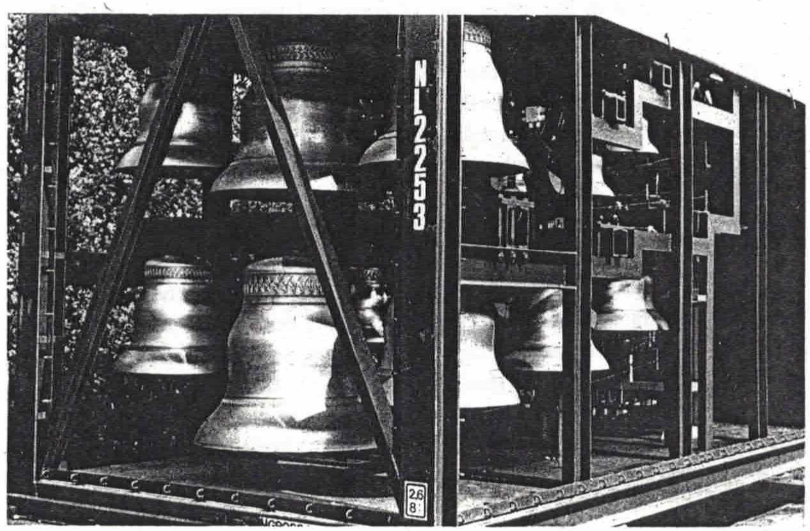

Figure 7.1.1. A carillon with major-third bells.

Although there was no systematic investigation during the various presentations of the major-third carillon, we can say that the conclusions of the investigations at the Institute for Perception Research of Technical University in Eindhoven, already described in Paragraph 3.5, were completely confirmed: the "average carillonneur" prefers the old trusted minor carillon, other musicians have a preference related to the key of the music played, while the layman has a clear preference for the major carillon.[106] We have to add that these

106. Houtsma and Tholen 1986.

are generalizations, although the extremes in opinion were wider for the carillonneurs than for the laymen. In other words, some carillonneurs reject the major carillon totally, whereas others react very enthusiastically. The extremes for laymen are much closer and are completely within the range of acceptance. But why this rejection and why this appreciation?

Starting with the last point, it seems that the unprejudiced layman especially appreciates that the pitch of the bell is much clearer. Moreover, he appreciates the shorter decay time, resulting in successive chords that intermingle less. Both aspects, of course, result from the properties of the major bell. First, this better discrimination of tones can be explained by a more fitting harmonic series. Secondly, a role is played by the fact that the difference tone between the fundamental C_5 and the third E_5 has become an octave below the hum note, whereas in the minor bell this is a major tenth.

There are other reasons why the major bell is perceived as less dissonant than the minor bell. This has nothing to do with the fact that the various intervals have more harmonic ratios, but presumably with the fact that the mutual intervals, at least in the lower part of the tonal spectrum, are usually larger than the so-called critical bandwidths in the ear.[107] In this context, it can be noted, amongst others, that the two closely neighbouring eleventh's F_6 (III-2/3) from the minor bell, partials that moreover are hardly a semitone away from the major tenth E_6 (II-4), have been shifted in the major-third bell to less disturbing positions, in particular to the diminished seventh B^{b}_{6}-, and to a position corresponding to the nominal C_6. In conclusion, everything combines to strengthen an unambiguous strike note or pitch.

The shorter decay time and, coupled to that, the reduced sonority that we have already mentioned in the previous chapter, result from the relatively high acoustic and material damping, that in turn, result from the higher f·D value respectively from the more complicated bell model. Apparently, this shorter duration is judged by the layman as an asset, as a chord is no longer disturbed by the resounding bells of a previous chord. During his playing on a minor carillon, of course, a professional carillonneur will take this phenomenon into account, as it is one of the elements which have become familiar to him during his study.

Obviously these clearer tones do not escape the carillonneur. But he attaches much less value to them as, in his opinion, they have less fascinating sounds. He is enthralled, over and over again, by the constant dissonances heard in the traditional bell, and apparently does not fully comprehend that he, as an educated musician, can dissect these dissonances moreover, he has become so familiar with them, thanks to his frequent carillon playing, that he can't even do without these dissonances. But the layman has not acquired these musical capabilities, either by study nor by frequent listening. For him, some carillonneurs are completely out of touch if they qualify the major bell in terms such as "synthetic", "electronic" etc. Although negative, these terms are a clear confirmation that the higher pureness and higher consonance of the major bell sound are indeed real, and do not exist in the imagination of the listener in the street and of those who appreciate the major carillon.

107. Rossing 1983, p. 85-86, 138-142.

In another respect also, the active carillonneur will be disappointed in his expectations of the major bell. Not only does he know better than everyone how a minor bell sounds, but more importantly he knows how the minor bell will react in certain compositions. He played a hundred times, for instance, a passage in a well-known piece of carillon music, in which the minor third of the traditional bell enforced just that one chord. But how he is taken in when the same passage is played on a major carillon. He is seized with fright, for although he could have predicted the fight between major and minor, he was mentally not at all prepared for this. In the opposite case, where one chord in that particular composition on minor bells was always dissonant - and how beloved this transition chord has become for him - how infinitely boring and empty it seems on those new, well-balanced major bells!

The problem is that an artist's career, formed so much by a particular musical instrument, has become dependent on that same instrument. The musician and the instrument are so tightly coupled and form such a balance that an outsider cannot disturb these, by modifying the instrument, without repercussion. Only by negating the new, that is a physical reality that seems not to be a musical instrument, can he survive in an artistic sense. It is as if he, the threatened musician, wants to prove the chemical law of Le Chatelier and Van 't Hoff - also known as the law of the least constraint - for music too: "If one of the factors of a system in balance is changed, then that reaction will proceed that counteracts the change."

Of course, only a few are so extreme in their rejection. Will they finally be right, in the same way as other new musical instruments were not successful? This is a possibility, though there is hardly any sign of this now. In this respect one could speak - as several carillonneurs have - of a totally new belfry instrument, that is, like the minor carillon, very suitable for particular situations, and in some circumstances should be the first choice. Realizing that the major carillon is less dissonant, has better recognizable tones, a shorter time of decay and a softer intonation etc., it seems to be the preferred instrument not only for low belfries, but also for accompanying brass. The latter has been proven several times, as was recently done at the great military tattoo at Breda in 1986! And we have not even mentioned the new possibilities seen in these new sounds by some enthusiastic composers.

There is still a final question to be answered, though it has nothing to do with practice. Is this new major carillon, so much appreciated by the unprejudiced layman, just a chance hit, originated in the search for the major-third bell, a bell that indeed was found, but for which in passing and not as a goal, some disturbing dissonances and an excessive sonority of the minor bell were removed? Were the desired major third and the dissonances - by chance disappeared - and tempered sonority together necessary for the success of the major carillon?[108] The future will show!

108. See also Lehr 1986 and Lehr 1987.

7.2 A variant of the first major-third bell

The previous paragraph was not only an attempt to determine what tonal differences exist between major and minor bells, but also amply discussed the fact that the layman does not attach much value to existing traditions, appreciating the smaller amount of sound and the much clearer tones of the major bell. In contrast, carillonneurs and those that have listened frequently to traditional carillons, have not succeeded in freeing themselves from the entrusted minor timbre, that has gained a value of beauty which seemed both absolute and irreplaceable. Does this bring doubt to the investigator, wondering if it is after all possible to make a major bell with the sonority of the minor bell? This question seemed even more justified, since at the quadrennial World Carillon Congress, in Ann Arbor (Mich., USA) in June 1986, this wish was expressed by some carillonneurs. So, why not a search for a richer sounding variant of the major-third bell, as this could resolve the misunderstanding that only one profile is possible for the major-third bell?

In designing the second major-third bell, no use was made of the computer, but instead of the knowledge that recently became available,owing to this apparatus, and especially of graphs for changes in profile and thickness that were very accurately computed (Figure 4.2.1-3). From these graphs it is easily seen that an outward movement of the waist indeed raises the third I-3 with respect to the nominal I-4. The graphs show that this effect is almost exclusively due to the outward movement of the outer side, a local increase in thickness, and not of the inner side. Apparently the same effect can be obtained by making a bulge on the outer side, leaving the inner side intact. The inner side could keep its traditional profile.

Secondly, the starting point for this newly-devised bell type was a traditional profile in which the third is as high as possible, as in the case of the bell profile that was used by the Royal Eijsbouts in 1955 for the carillon of, amongst others, Zeist (Section 3.5), a bell type for which the minor third already is about 50 cents sharp. This increase within the traditional pattern of the bell model permitted the limitation of extreme changes in profile as much as possible.

After some test models this led to the profile drawn in Figure 7.2.1. Figure 7.2.2 superimposes profiles of this new major third-bell, the first version of the major-third bell (Figure 6.3.1), and also the usual minor-third bell. Both versions of the major bell clearly demonstrate that more variation can occur in bell profiles than is usually assumed! Apparently this variation is not only found in the form of the profile, but also in the average thickness of the bell wall in relation to the dimensions of the bell, or, in other words, its f·D value. Soon it was found that, in contrast to the first major-third bell, there was from a viewpoint of tonal structure, no reason at all to increase this value above 200 m/s in this type of bell. Obviously, it was to be expected that the higher overtones, the partials above the nominal I-4, would not have the same position as in the first version of the major bell (Table 7.2.1).

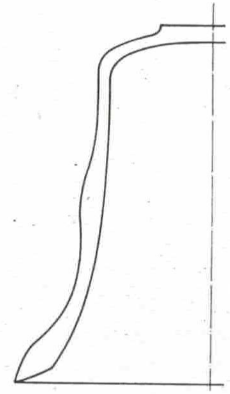

Figure 7.2.1. The second major-third bell.

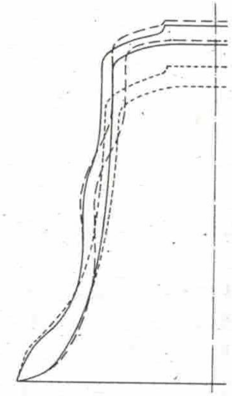

Figure 7.2.2. The minor-third bell and the first and second major-third bell.

partial	I-2	II-2	I-3	II-3	I-4	III-3	III-2	II-4	I-5	III-4	II-5	I-6
loudness	mf	f	f	mp	fff	p	p	mp	ff	p	p	f
major bell	C_4	C_5	E_5	G_5	C_6	$C^{\#}_6-$	D_6	E_6+	G_6	A^b_6	B_6+	C_7
	2 :	4 :	5 :	6 :	8 :	8.5 :	9 :	10.3 :	12 :	12.7 :	15.5 :	16
minor bell	C_4	C_5	E^b_5	G_5	C_6	F_6+	F_6-	E_6	G_6	A_6-	B_6	C_7+
	5 :	10 :	12 :	15 :	20 :	26.8 :	26.4 :	25 :	30 :	33 :	36 :	41

Table 7.2.1. The second major-third bell in comparison with the minor third bell.

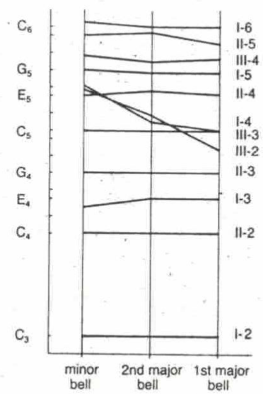

Figure 7.2.3. The partials of the minor-third bell and of the first and second major-third bell.

It is clear that the first two partials of the third group, III-2 and III-3, have returned to their positions above the nominal I-4. On the other hand, they are still under the major tenth II-4. Apparently this design is just between the minor bell and the first major bell, at least regarding the position of the partials, for which no conditions were set during the designing phase. This is visually presented in the graphs in Fig.7.2.3. But what about the result?

Owing to the fact that the profile of this second major-third bell is much closer to the classical bell profile, and to the fact that the f·D value has returned to its old entrusted value, it could be expected that the sound volume and the decay times would also be closer to tradition than for the first version of the major bell. Indeed, the difference is striking; it would be correct to conclude that apparently it is possible to cast a major-third carillon with richsounding, long-lasting bells, or one with drier and short-sounding bells. The choice is not only for those who play the carillon, but also, and not in the least part, for those who listen to it!

7.3 Other possible bell types

After the successful search for the major-third bell one may wonder whether other bell types are also within reach. One does not immediately have to think of the somewhat dubious seventh and ninth bells, but rather of bells in which such changes have been made in the mutual ratios between the partials that full emphasis can be given to music. Several of these suggestions have been gathered in Table 7.3.1 below. In the middle part of this table the ratio of the partial I-4 has been set to 4, and the other partials were computed to that ratio. On the right-hand side the most simple integers to describe these series have been computed.

bell type	I-2	II-2	I-3	II-3	I-4	most simple ratios
bell with harmonic partials	C_4 4/5	C_5 8/5	G_5 12/5	C_6 16/5	E_6 4	1 2 3 4 5
fifth bell	C_4 1	G_4 3/2	C_5 2	G_5 3	C_6 4	2 3 4 6 8
major octave bell	C_4 1	C_5 2	E_5 5/2	G_5 3	C_6 4	2 4 5 6 8
fourth-sixth bell	C_4 1	A_4 5/3	C_5 2	F_5 8/3	A_5 10/3	3 5 6 8 10
twelfth bell	G_3 2/3	C_5 2	E_5 5/2	G_5 3	C_6 4	4 12 15 18 24
minor-sixth bell	E_4 5/4	C_5 2	E_5 5/2	G_5 3	C_6 4	5 8 10 12 16
minor octave bell	C_4 1	C_5 2	E^b_5 12/5	G_5 3	C_6 4	5 10 12 15 20
major-sixth bell with diminished fundamental	E^b_4 6/5	B^b_4 9/5	E^b_5 12/5	G_5 3	C_6 4	6 9 12 15 20
major-sixth bell	E^b_4 6/5	C_5 2	E^b_5 12/5	G_5 3	C_6 4	6 10 12 15 20
minor-sixth bell with diminished fundamental	E_4 5/4	A_4 5/3	E_5 5/2	A_5 10/3	C_6 4	15 20 30 40 48
octave bell with diminished fundamental	C_4 1	B^b_4 9/5	D_5 9/4	F_5 27/10	C_6 4	20 36 45 54 80

Table 7.3.1. Existing bell types and some suggestions for other types

A further explanation to this Table 7.3.1 is the following. The octave bell with minor third is the usual carillon bell, and to a lesser extent the swinging bell of Western Europe. The last bell listed in the table, the octave bell with diminished fundamental is a bell that became traditional, especially in Russia. This shows that a maximal consonance does not automatically result in an equally high acceptance. On the contrary, even dissonant bells can be appreciated after habituation! Besides the traditional minor bell and this Russian bell, the profile for a minor-sixth bell is also known to the Royal Eijsbouts bellfoundry.

The octave bell with a major third has been available since March 1985. The quest for this bell was one of the subjects of this study. In connection to the concomitantly more harmonic character, as the hum note can start with 2 whereas for the minor octave bell it should be 5, one could search for a perfectly harmonically-structured bell. This does not seem to be too unrealistic, as also in the partials above I-4 such an arrangement appears possible.

To avoid these thoughts to merely qualitative considerations, we could aim to compute the degree of consonance of each bell chord. Several computational models are available for this. The oldest is provided by Euler, the mathematician who we already know from his "Tentamen de sono campanarum". In his "Tentamen novae theoriae musicae", published in Leningrad (St.Petersburg) in 1739, he introduced the conception of "gradus suavitatis", the degree of sweetness of a chord.[109] More modern computational models were set up by Plomp and Levelt in The Netherlands and Kameoka and Kuriyagawa in Japan.[110] Without further discussing these theories, given that the simple ratios are central to them, the arrangement in Table 7.3.1 is more or less according to a decreasing degree of consonance.

Obviously it is not an easy task to indicate which bell chords should be realized. Not only musical considerations play a role, but probably campanological restrictions too. On the one hand, the potentialities seem almost unlimited, but sometimes these limits seem very close! Although test models have shown, for example, that it is even possible to position I-2 above II-2 - so I-2 can no longer be considered as the hum note - we know nevertheless, of the difficulties of maneuvring a partial within one and the same group. Based on many experiments at Royal Eijsbouts, it must be concluded that the shifting of partials in relation to profile changes must be divided into two groups:

1. Large shifts between I-2, II-2, group I without I-2, group II without II-2, group III, group IV etc.
2. Small shifts within the groups themselves, i.e. in group I without I-2, group II without II-2, group III, group IV etc.

It must be noted that the partials I-2 and II-2 apparently operate completely independently, reacting to profile changes, as it were, like an individual group on their own. This phenomenon was confirmed again and again in experiments. From this point of view, the introduction of a group zero by Rossing and Perrin can surely be upheld (section 3.3).[111]

109. Euler 1739. See also Fokker 1945, p. 42-43.
110. Plomp and Levelt 1965; Kameoka and Kuriyagawa, 1969. See also Vos 1986, p. 59-95

It should be added emphatically that fluent profile lines were a condition for those test models. So, it is not excluded that these rules can be infringed when micro changes are made in the profile, very local changes giving the profile several transitions from concave to convex, and even sharp discontinuities. This is still a subject of further research.[112] For the time being, we restrict ourselves to the rules that apply to macro changes and that result in fluent profile lines.

It is from these rules that Table 7.3.1 resulted. The tonal spectrum was constantly compared to these rules in order not to present possibilities that a priori seem unattainable. For example, it seems completely impossible to transform the series I-3 to I-9 that can be described for the minor bell as (Table 3.3.1):

$$E^b_4 - C_5 - G_5 - C_6+ - F_6 - A_6 - C_7$$

by using a change of model into:

$$E^b_4 - C_5 - E^b_5 - C_6 - E^b_6 - G_6 - C_7.$$

This is because it seems hardly possible to keep the position of four partials identical, and simultaneously drastically shift the positions of the intermediate partials, at least if no micro changes are to be made in the profile.

The possible types of bell chords presented in Table 7.3.1 do not imply that, if these bells were to be realized, the results obtained would be of direct musical importance. Hardly anything is known of the effects on volumes and decay times of the individual partials. Nevertheless, it is a factor that should be taken into account, in view of the two versions of the major-third bell, so different in timbre. Therefore, the above-mentioned table is primarily to be seen as an incentive - albeit a very essential one - for further investigation of bell profiles, to give an opportunity, both to the future musicians and to their future audience in the streets, to enjoy the ancient music of bells in a new variety of sounds and tones.

112. Schoofs 1987.

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